BTP SIES ON $\mathbb{R}_+ \times \mathbb{R}^d$: ULTRA REGULAR BTRW SIES LIMITS SOLUTIONS, THE K-MARTINGALE APPROACH, AND FOURTH ORDER SPDES LINKS

HASSAN ALLOUBA

ABSTRACT. We delve deeper into the compelling regularizing effect of the Brownian-time Brownian motion density, $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$, on the space-time-white-noise-driven stochastic integral equation we call BTP SIE:

$$(0.1) \quad U(t,x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y)dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} a(U(s,y)) \mathscr{W}(ds \times dy),$$

which we recently introduced in [1]. In sharp contrast to traditional second order heat-operator-based SPDEs—whose real-valued mild solutions are confined to $d=1$—we prove the existence of L^p-bounded ultra regular solutions of (0.1) in $1 \leq d \leq 3$, under both less than Lipschitz and Lipschitz conditions on a (uniqueness holds under the later). In space, we show a remarkable nearly local Lipschitz regularity for $d=1,2$; and we prove nearly local Hölder $1/2$ regularity in $d=3$. In time, our solutions are locally γ-Hölder continuous with exponent $\gamma \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$. To investigate (0.1) under less than Lipschitz conditions on a, we (a) introduce the Brownian-time random walk (BTRW)—a special case of lattice processes we call BTCs—and we use it to formulate the spatial lattice version of (0.1) (BTRW SIE); and (b) develop a delicate variant of Stroock-Varadhan martingale approach, the K-martingale approach, tailor-made for a wide variety of kernel SIEs including (0.1) and the mild forms of many SPDEs of different orders on the lattice. Solutions to (0.1) are defined as limits of the BTRW SIEs. Along the way, we prove interesting aspects of BTRW, including a fourth order differential-difference equation connection. We also include a direct proof of existence, pathwise uniqueness, and the same Hölder regularity for (0.1), without discretization, in the Lipschitz case. The SIE (0.1) is intimately connected to intriguing fourth order SPDEs in two ways. First, we show that (0.1) is connected to the diagonals of a new unconventional fourth order SPDE we call parametrized BTP SPDE. Second, replacing $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ by the intimately connected kernel of our recently-introduced imaginary-Brownian-time-Brownian-angle process (IBTBAP), (0.1) becomes the mild form of a Kuramoto-Sivashinsky (KS) SPDE with linearized PDE part. Ideas and tools developed here are adapted in separate papers to give an entirely new approach, via our explicit IBTBAP representation, to many linear and nonlinear KS-type SPDEs in multi-spatial dimensions.

1. Introduction and statement of results

1.1. **Motivation.** A fascinating aspect of the Brownian-time processes (BTPs) we introduced in [9, 8] is the rich interplay between them and fourth order PDEs. On one hand, BTPs solve non-Markovian (memory preserving) fourth order PDEs involving a positive bi-Laplacian that is coupled with a time-scaled positive Laplacian so as to produce smooth solutions, for all times and all spatial dimensions. The canonical such deterministic PDE is

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\Delta u_0}{\sqrt{8\pi t}} + \dfrac{1}{8}\Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = u_0(x); & x \in \mathbb{R}^d. \end{cases}$$

On the other hand, tweaking the BTPs a little by running the Brownian-time on the imaginary axis and adding a Brownian "angle" we obtain the imaginary-Brownian-time-Brownian-angle process (IBTBAP) [2]. The IBTBAP in turn gives a probabilistically inspired representation for the solution of a linearized version of the prominent fourth order Kuramoto-Sivashinsky (KS) PDE of modern applied mathematics

$$(1.2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{8}\Delta^2 u - \dfrac{1}{2}\Delta u - \dfrac{1}{2}u, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

as we showed in [2]. Because of the intimate relation between BTPs and the IBT-BAP, their kernels have a similar regularizing effect on solutions to their respective PDEs. This is despite the fact that BTP PDEs involve the positive bi-Laplacian while the KS PDE contains the more traditional well behaved negative bi-Laplacian. It is at least as intriguing to study BTP and BTP-type stochastic equations driven by additive and multiplicative space-time white noise. First, we show here that these equations ((1.3) below) have ultra regular solutions on $\mathbb{R}_+ \times \mathbb{R}^d$. In space, we show a rather remarkable and initially-surprising nearly local Lipschitz regularity for $d = 1, 2$; and we prove nearly local Hölder $1/2$ regularity in $d = 3$. This is remarkable because the BTP kernel is able, in $d = 1, 2$, to spatially regularize such solutions beyond the traditional Hölder-$1/2$ spatial regularity of the underlying Brownian sheet corresponding to the driving space-time white noise. This degree of smoothness is unprecedented for space-time white noise driven kernel equations or their corresponding SPDEs; and the BTP SIE is thus the first such example. In time, our solutions are locally γ-Hölder continuous solutions with dimension-dependent exponent $\gamma \in \left(0, \frac{4-d}{8}\right)$ for $1 \le d \le 3$. This is in sharp contrast to traditional second order reaction-diffusion (RD) and other heat-operator-based SPDEs driven by space-time white noise, whose fundamental kernel is the Brownian motion density and whose real-valued mild solutions are confined to the case $d = 1$. This sharp contrast in regularity is summarized in Table 3.1. In this regard, the dichotomy between the rougher paths of BTBMs as compared to standard BMs on the one hand (quartic vs. quadratic variations) and the stronger regularizing properties of the BTBM density vs. the BM one on the other hand is certainly another interesting point to make. Second, due to the intimate BTPs-IBTBAP connection; and just as we showed in the deterministic PDEs case [2, 8, 9]—where methods from BTPs PDEs were adapted to prove results for the linearized KS PDE (1.2) *in all dimensions*—our study here is very helpful in our related investigation of KS-type SPDEs which we treat in separate papers [3, 4, 6]. The methods presented here

and in [2] are adapted and generalized in [3, 4, 6] to give an entirely new approach, in terms of the IBTBAP kernel and related probabilistically motivated concepts, to the SPDE version of these famous fourth order applied mathematics PDEs in $d = 1, 2, 3$. Here, it is noteworthy that even the existence of the KS semigroup is not known analytically for $d > 1$. The regularity of the SPDEs in [3, 4, 6] is very similar to this paper's result.

Before precisely stating the results of this article, it is necessary to recall as well as introduce the various recent and new constructions and technical ingredients involved. Recalling very briefly, a BTP is a process $X^x(|B_t|)$ in which X^x is a Markov process starting at $x \in \mathbb{R}^d$ and B_t is an independent one dimensional BM starting at 0. A Brownian-time Brownian motion (BTBM) is a BTP in which X^x is also a Brownian motion. From a probabilistic point of view, BTPs form a unifying canonical class for many different exciting processes like the iterated Brownian motion (IBM) of Burdzy (a process with fourth order properties: fourth variation [15, 16] and fourth-order PDEs [9, 8, 21]) and the Brownian-snake of Le Gall (a second-order process by virtue of its connection to the nonlinear PDE $\Delta u = u^2$ as in [28]). BTPs also include many additional new and quite interesting processes, which we are currently investigating in several directions. With the exception of the Markov snake, BTPs fall outside the classical theory of Markov, Gaussian, or semimartingale processes. From a PDE perspective, BTPs give rise to regular ($C^{1,4}$) solutions to new different fourth order PDEs—like (1.1)—that involve the positive bi-Laplacian, for all times and in all spatial dimensions. The intrigue comes not only from directly connecting these BTPs to PDEs despite the lack of classical properties for the underlying processes (non-semimartingales, non-Marokvian, and non-Gaussian), but also from the fact that typical positive bi-Laplacian PDEs are not well behaved. Nevertheless, BTPs lead to equations in which the positive bi-Laplacian is coupled in a very specific way—dictated by the BTP probability density function—with a time-scaled Laplacian acting on the smooth initial data whose smoothing effect gets arbitrarily large as time $t \searrow 0$ and fades away as $t \nearrow \infty$ at the rate of $1/\sqrt{8\pi t}$; and the BTP solutions to these BTP PDEs are eternally highly regular.

In this paper, we continue our study of the regularizing effect the BTBM kernel has on the space-time white noise driven BTBM (or BTP) stochastic integral equation (BTP SIE), first introduced in [1]:

$$(1.3) \qquad U(t,x) = \int_{\mathbb{R}^d} \mathbb{K}^{\text{BTBM}^d}_{t;x,y} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}^{\text{BTBM}^d}_{t-s;x,y} a(U(s,y)) \mathscr{W}(ds \times dy)$$

where $\mathscr{W}$ is the white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and $\mathbb{K}^{\text{BTBM}^d}_{t;x,y}$ is the density of a Brownian-time Brownian motion given by:

$$(1.4) \qquad \mathbb{K}^{\text{BTBM}^d}_{t;x,y} = 2 \int_0^\infty K^{\text{BM}^d}_{s;x,y} K^{\text{BM}}_{t;0,s} ds$$

with $K^{\text{BM}^d}_{s;x,y} = \frac{e^{-|x-y|^2/2s}}{(2\pi s)^{d/2}}$ and $K^{\text{BM}}_{t;0,s} = \frac{e^{-s^2/2t}}{\sqrt{2\pi t}}$. We call (1.3) BTP SIE (or BTBM SIE) since it is expressed in terms of the density (or kernel) of some BTP, a BTBM in this case of (1.3). We denote our BTP SIE (1.3) by $e^{\text{SIE}}_{\text{BTP}}(a, u_0)$.

Equation $e^{\text{SIE}}_{\text{BTP}}(a, u_0)$ has several rather interesting features that motivate and lead to the tools and the results of this article. Here are some:

(1) $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ is a kernel stochastic integral equation whose kernel $\mathbb{K}^{\mathrm{BTBM}^d}_{t;x,y}$ is the density of a compelling stochastic process that solves an unconventional memory-preserving fourth order PDE (1.1) and that does not belong to any of the classical families of processes. In order to treat (1.3) under less than Lipschitz conditions on a

- we introduce a new class of processes we call Brownian-time chains (BTCs), which is of interest on its own and it contains the Brownian-time random walk (BTRW) as a special case;
- we formulate (1.3) on discrete spatial lattices in terms of the BTRW density, and we define solutions to $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ as limits of the lattice models (as we did in [7] for the simpler RD SPDEs using standard RW); and
- we develop a new variant of Stroock-Varadhan famous martingale approach to stochastic equations [32], which is tailor-made for a large class of kernel SIEs (including those that are the mild form of SPDEs of second and fourth orders) through their discretized versions and we call it the K-martingale approach (see the discussion in Section 1.4).

(2) $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ has Hölder continuous paths in time and space with temporal Hölder exponent $\gamma_t \in (0, \frac{4-d}{8})$ for spatial dimensions $d = 1, 2, 3$, despite being driven by a space-time white noise. In addition, it has a striking nearly local Lipschitz spatial regularity for $d = 1, 2$, and nearly local Hölder $1/2$ spatial regularity for $d = 3$. This is in stark contrast to second order white noise driven RD SPDEs whose real-valued mild solutions are confined to the one spatial dimension case ($d = 1$).

(3) $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ has an intriguing connection to fourth oder SPDEs both directly and indirectly. Just as the deterministic ($a \equiv 0$) version of (1.3) is linked to unconventional fourth order PDEs like (1.1) (see e.g., [9, 8]), we link $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ to the diagonals of a new unconventional fourth order SPDE that we call parametrized BTP SPDE. On the other hand, by using the IBTBAP's kernel to replace $\mathbb{K}^{\mathrm{BTBM}^d}_{t;x,y}$ in (1.3)—as we did in the deterministic case in [2]—we obtain a mild formulation of KS-type SPDEs as in [3, 4, 6, 1] (see Section 1.6 below for a brief overview). This allows for the treatment of several important nonlinear applied fourth order SPDEs in multi-spatial dimensions, including the Swift-Hohenberg (SH) and variants of the KS SPDEs (in [4, 6] and in planned followup articles). Due to the intimate relation between the BTBM and the IBTBAP, our study of $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ here sheds light on how to approach these major SPDEs using a probabilistically inspired representation; and $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ may be looked at as a cousin of and a companion model for such SPDEs.

In the BTP stochastic equations setting, the BTP SIE (1.3) is different from the stochastic version of (1.1), viz. the BTP SPDE

$$(1.5) \qquad \begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\Delta u_0}{\sqrt{8\pi t}} + \dfrac{1}{8}\Delta^2 U + a(U)\dfrac{\partial^2 W}{\partial t \partial x}, & (t, x) \in (0, \infty) \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

It is $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ that has regular solutions in $d > 1$—capturing the smoothing effect of the BTBM density in the stochastic setting—and it is the equation intimately related to the KS, SH, and other important fourth order SPDEs of modern applied

mathematics. We therefore focus the bulk of our investigation and our main results in this article on (1.3). In Section 1.6 we discuss further the links between $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ and fourth order SPDEs of KS type as well as the new parametrized BTP SPDE, relative to which (1.5) may be thought of as a degenerate version.

In [1] we considered the additive noise case $a \equiv 1$ for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$, and we proved the existence of a pathwise unique continuous BTP SIE solution $U(t, x)$ for $x \in \mathbb{R}^d$ and $1 \le d \le 3$, such that

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_{\mathbb{P}} |U(t, x)|^{2p} \le C \left[1 + t^{\frac{(4-d)p}{4}} \right]; \quad t > 0, \ 1 \le d \le 3, \ p \ge 1.$$

In this paper, we prove existence and finer Hölder regularity results for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ under the following less than Lipschitz conditions on the Borel-measurable diffusion coefficient a:

(NLip) $\quad \begin{cases} (a) & a(u) \text{ is continuous in } u, \quad u \in \mathbb{R}; \\ (b) & a^2(u) \le C(1 + u^2), \quad u \in \mathbb{R}; \\ (c) & u_0 \in \mathrm{C}^2(\mathbb{R}^d; \mathbb{R}) \text{ and nonrandom and bounded with } D_{ij} u_0 \\ & \text{bounded and Hölder continuous } \forall \ 1 \le i, j \le d \end{cases}$

as well as uniqueness for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ under an added Lipschitz condition on a.

We now detail the structure of the remainder of this article. The rest of Section 1 provides the setting then states the main result, Section 2 contains the proofs of several lemmas and results used in the proof of the main result for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ and it contains a proof of fourth order SPDEs-BTP SIEs connection on the lattice, Section 3 contains additional concluding remarks. More specifically, from Section 1.2 to Section 1.4; we give all the ingredients necessary to state our main existence and regularity result, given in Section 1.5. In Section 1.2 we introduce a new class of discrete-valued processes that we call Brownian-time chains (BTCs). BTCs are the discretized versions of our BTPs in [9, 8]. Of particular interest here is the special case of Brownian-time random walk (BTRW), which we define and link to the lattice version of (1.1) (Lemma 1.1). In Section 1.3, we use the density of the BTRW to give a spatially-discretized formulation (1.11) of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$, which we call BTRW SIE, and we define two notions of solutions to the lattice model: direct solutions and limit solutions (from a finite truncation of the lattice to the whole lattice). These solutions (both direct and limit) are then used to define two types of BTRW SIEs limit solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ (direct limit solutions and double limit solutions), as the size of the lattice mesh shrinks to zero. We introduce our K-martingale approach to kernel SIEs as $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ in Section 1.4. As stated above, it is a delicate variant of the well known, and by now classic, martingale problem approach of Stroock and Varadhan to SDEs. A key advantage of the K-martingale approach is that it is a unified framework in which the existence and uniqueness of many kernel stochastic integral equations, which are the mild formulation for many SPDEs, may be treated under less than Lipschitz conditions, using only variants of the kernel formulation of the underlying equation. This includes SPDEs of different orders (second and fourth), so long as the corresponding spatially-discretized kernel (or density) satisfies Kolmogorov-type bounds on its temporal and spatial differences. In essence, what the K-martingale approach implies is that if the kernel in the lattice model is nice enough for the lattice model to converge as the lattice mesh shrinks to zero (under appropriate assumptions on a), then it is nice enough

to guarantee a solution for the lattice model. We use it here to prove the existence of solutions to (1.3) under the conditions (NLip), but just as with the Stroock-Varadhan method, it can handle uniqueness as well. Our K-martingale approach starts by constructing an auxiliary problem to a truncated lattice version of (1.3), for which the existence of solutions implies solutions existence for the truncated lattice model. We then formulate a martingale problem equivalent to the auxilliary problem (the K-martingale problem). Section 1.5 contains the main existence and dimension-dependent regularity result for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. The upshot is that under both Lipschitz and non-Lipschitz conditions on a ((Lip) and (NLip)) there exist BTRW SIE limit solutions (defined in Section 1.3). These solutions are extracted as weak limits of a system of BTRW SIEs (the type of limit solution depends on the conditions: direct limit solution for the Lipschitz case and double limits solutions for the less-than-Lipschitz case). This discretization method is similar in spirit to our discretization approach for the simpler second order reaction-diffusion (RD) SPDEs in [11, 7]. We prove here that these limit solutions are temporally locally Hölder continuous with Hölder exponent $\gamma \in (0, \frac{4-d}{8})$ for spatial dimensions $d = 1, 2, 3$. Spatially, they have an impressive nearly local Lipschitz regularity for $d = 1, 2$, and nearly local Hölder $1/2$ regularity in $d = 3$. Here we observe in passing that—roughly speaking—the paths of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ in $d = 1$ are $3/2$ times as smooth as the RD SPDE paths in $d = 1$, in $d = 2$ the BTP SIE is as smooth as an RD SPDE in $d = 1$, and in $d = 3$ our BTP SIE is half as smooth as an RD SPDE in $d = 1$. Also, for $d = 2, 3$, the spatial regularity is roughly four times the temporal one, and in $d = 1$ the spatial regularity is maximized at a near Lipschitz vs near Hölder $3/8$ in time. Section 1.6 gives the BTP SIEs-SPDEs connections. It gives a brief look into the indirect KS-type SPDEs connection—via the IBTBAP kernel—to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. It also provides the intriguing connection of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ to the parametrized BTP SPDE on the lattice. In Section 2 we prove different BTRW and BTBM estimates—introducing along the way the notion of 2-Brownain-times RW and BM; some BTRW SIEs estimates necessary for regularity and tightness; the K-martingale result; and the main existence, uniqueness, and regularity result for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ (Theorem 1.2). Some interesting related results are discussed and/or proved in Appendices A, B, and C. Appendix A gives the proof of the BTRW fourth order differential-difference equation on lattices in Lemma 1.1. Appendix B contains a direct proof of the existence, uniqueness, and regularity for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ under Lipschitz conditions without discretization via an iterative-type argument. Appendix C contains more on the BTP SIEs-SPDEs connections on lattices.

Remark 1.1. Throughout this article, whenever needed, we will assume that our filtrations satisfy the usual conditions without explicitly stating so. As is customary, C will denote a constant that may change its value from one line to the next. We will denote the euclidean distance on d-dimensional spaces by $|\cdot|$. By the set $\mathbb{T}$ we always mean the time interval $\mathbb{T} := [0, T]$ for some fixed but arbitrary $T > 0$.

1.2. **Brownian-time random walk and chains on the lattice.** In [11, 7], standard continuous-time random walks on a sequence of refining spatial lattices

$$\left\{ \mathbb{X}_n^d := \prod_{i=1}^{d} \{\ldots, -2\delta_n, -\delta_n, 0, \delta_n, 2\delta_n, \ldots\} = \delta_n \mathbb{Z}^d \right\}_{n \geq 1}$$

(with the step size $\delta_n \searrow 0$ as $n \to \infty$) played a crucial role—through their densities—in obtaining our results for second order RD SPDEs. Here, in the fourth order Brownian-time setting, that role is played by Brownian-time random walks on $\mathbb{X}_n^d$:

$$(1.6) \qquad \mathbb{S}_{B,\delta_n}^x(t) := S_{\delta_n}^x\left(|B_t|\right); \quad 0 \le t < \infty, x \in \mathbb{X}_n^d$$

where $S_{\delta_n}^x(t)$ is a standard d-dimensional continuous-time symmetric RW starting from $x \in \mathbb{X}_n^d$ and B is an independent one-dimensional BM starting at 0. The subscript δ_n in (1.6) is to remind us that the lattice step size is δ_n in each of the d directions. It is then clear that the transition probability (density) $\mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d}$ of the BTRW $\mathbb{S}_{B,\delta_n}^x(t)$ on $\mathbb{X}_n^d$ is given by

$$(1.7) \qquad \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} = 2 \int_0^\infty K_{s;x,y}^{\mathrm{RW}_{\delta_n}^d} K_{t;0,s}^{\mathrm{BM}} ds; \quad 0 < t < \infty, \ x,y \in \mathbb{X}_n^d$$

with $K_{t;0,s}^{\mathrm{BM}} = \left(1/\sqrt{2\pi t}\right) \exp\left\{-s^2/2t\right\}$ and $K_{t;x,y}^{\mathrm{RW}_{\delta_n}^d}$ is the continuous-time random walk transition density starting at $x \in \mathbb{X}_n^d$ and going to $y \in \mathbb{X}_n^d$ in time t, in which the times between transitions are exponentially distributed with mean δ_n^{2d}. Throughout this article, $\mathbb{K}_{t,x}^{\mathrm{BTRW}_{\delta_n}^d} := \mathbb{K}_{t;x,0}^{\mathrm{BTRW}_{\delta_n}^d}$ (with a similar convention for all transition densities). I.e., $K_{t;x}^{\mathrm{RW}_{\delta_n}^d}$ is the fundamental solution to the deterministic heat equation on the lattice $\mathbb{X}_n^d$:

$$(1.8) \qquad \frac{du_n^x(t)}{dt} = \frac{1}{2}\Delta_n u_n^x(t); \ \ (t,x) \in (0,\infty) \times \mathbb{X}_n^d$$

where $\mathscr{A}_n := \Delta_n/2$ is the generator of the RW $S_{\delta_n}^x(t)$ on $\mathbb{X}_n^d$.

By mimicking our proof of Theorem 0.1 in [9] (see the proof in Appendix A), we easily get the following fourth order differential-difference equation connection to BTRW:

Lemma 1.1 (BTRW's DDE)**.** *Let* $u_n^x(t) = \mathbb{E}\left[u_0\left(\mathbb{S}_{B,\delta_n}^x(t)\right)\right]$ *with* u_0 *as in* (NLip). *Then* u_n *solves the following fourth order differential-difference equation (DDE) on* $\mathbb{R}_+ \times \mathbb{X}_n^d$:

$$(1.9) \qquad \begin{cases} \dfrac{du_n^x(t)}{dt} = \dfrac{\Delta_n u_0(x)}{\sqrt{8\pi t}} + \dfrac{1}{8}\Delta_n^2 u_n^x(t), & (t,x) \in (0,\infty) \times \mathbb{X}_n^d \\ u_n^x(0) = u_0(x), & x \in \mathbb{X}_n^d \end{cases}$$

and $\mathbb{K}_{t,x}^{\mathrm{BTRW}_{\delta_n}^d}$ *solves* (1.9) *on* $[0,\infty) \times \mathbb{X}_n^d$, *with*

$$(1.10) \qquad u_0(x) = \mathbb{K}_{0;x}^{\mathrm{BTRW}_{\delta_n}^d} = K_{0;x}^{\mathrm{RW}_{\delta_n}^d} = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

BTRWs are the discretized version of our BTBM in [9, 8]. They belong to a large and new class of discrete-valued processes which we now introduce. Suppressing the n in the lattices $\mathbb{X}_n^d$, let B be a one-dimensional Brownian motion starting at 0 and let D^x be an independent d-dimensional $\mathbb{X}^d$-valued continuous-time Markov chain starting at x, both defined on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$. We call the process $\mathbb{D}_B^x(t) := D^x(|B_t|)$ a Brownian-time chain (BTC). A BTRW is a special case of BTCs in which D^x is a continuous-time random walk. Excursions-based Brownian-time chains (EBTCs) are obtained from BTCs by breaking up the path

of $|B_t|$ into excursion intervals—maximal intervals (r, s) of time on which $|B_t| > 0$—and, on each such interval, we pick an independent copy of the Markov chain D^x from a finite or an infinite collection. BTCs and EBTCs may be regarded as canonical constructions to some quite interesting new processes:

(1) Markov snake chain: when $|B_t|$ increases we pick a new chain D^x, we denote this process by $\mathbb{D}^x_{B,SC}(t)$.

(2) k-EBTCs: let $D^{x,1}, \ldots, D^{x,k}$ be independent copies of D^x starting from point $x \in \mathbb{X}^d$. On each $|B_t|$ excursion interval, use one of the copies chosen at random. we denote such a process by $\mathbb{D}^{x,k}_{B,e}(t)$. When $k = 2$ and D^x is a random walk on $\mathbb{X}^d$ this is a discrete version of Burdzy's IBM. When $k = 1$ we obtain a BTC.

(3) ∞-EBTCs: we use an independent copy of D^x on each $|B_t|$ excursion interval. This is the $k \to \infty$ of (2), It is intermediate between (1) and (2). Here, we go forward to a new independent chain only after $|B_t|$ reaches 0. This process is denoted by $\mathbb{D}^{x,\infty}_{B,e}(t)$.

1.3. BTRW SIEs and their limits solutions to BTP SIEs.
The crucial role of the BTRW density in our approach to the BTP SIE (1.3) becomes even clearer from the following definition of our approximating spatially-discretized equations:

Definition 1.1 (BTRW SIEs). By the BTRW SIEs associated with the BTP SIE $e^{\text{SIE}}_{\text{BTP}}(a, u_0)$ we mean the system $\left\{ e^{\text{SIE}}_{\text{BTRW}}(a, u_0, n) \right\}_{n=1}^{\infty}$ of spatially-discretized stochastic integral equations on $\mathbb{R}_+ \times \mathbb{X}^d_n$ given by

$$(1.11) \qquad \tilde{U}^x_n(t) = \sum_{y \in \mathbb{X}^d_n} \mathbb{K}^{\text{BTRW}^d_{\delta_n}}_{t;x,y} u_0(y) + \sum_{y \in \mathbb{X}^d_n} \int_0^t \mathbb{K}^{\text{BTRW}^d_{\delta_n}}_{t-s;x,y} a(\tilde{U}^y_n(s)) \frac{dW^y_n(s)}{\delta^{d/2}_n},$$

where the BTRW density is given by (1.7). For each $n \in \mathbb{N}$, we think of $W^x_n(t)$ as a sequence of independent standard Brownian motions indexed by the set $\mathbb{X}^d_n$ (independence within the same lattice). We also assume that if $m \neq n$ and $x \in \mathbb{X}^d_m \cap \mathbb{X}^d_n$ then $W^x_m(t) = W^x_n(t)$, and if $n > m$ and $x \in \mathbb{X}^d_n \setminus \mathbb{X}^d_m$ then $W^x_m(t) = 0$.

Notation 1.1. We will denote the deterministic and the random parts of (1.11) by $\tilde{U}^x_{n,D}(t)$ and $\tilde{U}^x_{n,R}(t)$ (or $\tilde{U}^x_D(t)$ and $\tilde{U}^x_R(t)$ when we suppress the dependence on n), respectively, whenever convenient.

We define two types of solutions to BTRW SIEs: direct solutions and limit solutions.

Definition 1.2 (Direct BTRW SIE Solutions). A direct solution to the BTRW SIE system $\left\{ e^{\text{SIE}}_{\text{BTRW}}(a, u_0, n) \right\}_{n=1}^{\infty}$ on $\mathbb{R}_+ \times \mathbb{X}^d_n$ with respect to the Brownian (in t) system $\{W^x_n(t)\}_{(n,x) \in \mathbb{N} \times \mathbb{X}^d_n}$ on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ is a sequence of real-valued processes $\left\{ \tilde{U}_n \right\}_{n=1}^{\infty}$ with continuous sample paths in t for each fixed $x \in \mathbb{X}^d_n$ and $n \in \mathbb{N}$ such that, for every $(n, x) \in \mathbb{N} \times \mathbb{X}^d_n$, $\tilde{U}^x_n(t)$ is $\mathscr{F}_t$-adapted, and equation (1.11) holds $\mathbb{P}$-a.s. A solution is said to be strong if $\{W^x_n(t)\}_{(n,x) \in \mathbb{N} \times \mathbb{X}^d_n}$ and $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ are fixed a priori; and with

$$(1.12) \qquad \mathscr{F}_t = \sigma \left\{ \sigma \left(W^x_n(s); 0 \leq s \leq t, x \in \mathbb{X}^d_n, n \in \mathbb{N} \right) \cup \mathscr{N} \right\}; \quad t \in \mathbb{R}_+,$$

where $\mathscr{N}$ is the collection of null sets

$$\{O : \exists\, G \in \mathscr{G}, O \subseteq G \text{ and } \mathbb{P}(G) = 0\}$$

and where

$$\mathscr{G} = \sigma \left(\bigcup_{t \geq 0} \sigma \left(W_n^x(s); 0 \leq s \leq t, x \in \mathbb{X}_n^d, n \in \mathbb{N} \right) \right).$$

A solution is termed weak if we are free to choose $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ and the Brownian system on it and without requiring $\mathscr{F}_t$ to satisfy (1.12). Replacing $\mathbb{R}_+$ with $\mathbb{T} := [0, T]$—for some $T > 0$ in the above, we get the definition of a solution to the BTRW SIE system $\left\{ e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n) \right\}_{n=1}^{\infty}$ on $\mathbb{T} \times \mathbb{R}^d$.

The next type of BTRW SIE solutions we define is the first step in our K-martingale approach of Section 1.4. By first reducing $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ to the simpler finite dimensional noise setting, it takes full advantage of the notion of BTRW SIEs limit solutions to BTP SIEs.

Definition 1.3 (Limit BTRW SIE Solutions)**.** Let $l \in \mathbb{N}$. By the l-truncated BTRW SIE on $\mathbb{R}_+ \times \mathbb{X}_n^d$ we mean the BTRW SIE obtained from (1.11) by restricting the sum in the stochastic term to the finite d-dimensional lattice $\mathbb{X}_{n,l}^d := \mathbb{X}_n^d \cap \left\{ [-l, l]^d; l \in \mathbb{N} \right\}$ and leaving unchanged the deterministic term $\tilde{U}_{n,D}^x(t)$:

$$(1.13) \qquad \tilde{U}_{n,l}^x(t) = \begin{cases} \tilde{U}_{n,D}^x(t) & + \displaystyle\sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n, s, t}^{x,y} \left(\tilde{U}_{n,l}^y(s) \right) dW_n^y(s); x \in \mathbb{X}_{n,l}^d, \\ \tilde{U}_{n,D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d \end{cases}$$

where

$$\kappa_{\delta_n, s, t}^{x,y} \left(\tilde{U}_{n,l}^y(r) \right) := \frac{\mathbb{K}_{t-s; x, y}^{\mathrm{BTRW}_{\delta_n}^d}}{\delta_n^{d/2}} a(\tilde{U}_{n,l}^y(r)), \quad \forall r, s < t.$$

We denote (1.13) by $e_{\mathrm{BTRW}}^{\mathrm{t\text{-}SIE}}(a, u_0, n, l)$. Fix $n \in \mathbb{N}$, a solution to the system of truncated BTRW SIEs $\left\{ e_{\mathrm{BTRW}}^{\mathrm{t\text{-}SIE}}(a, u_0, n, l) \right\}_{l=1}^{\infty}$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$ with respect to the Brownian (in t) system $\{W_n^x(t)\}_{x \in \mathbb{X}_n^d}$ on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ is a sequence of real-valued processes $\left\{ \tilde{U}_{n,l} \right\}_{l \in \mathbb{N}}$ with continuous sample paths in t for each fixed $x \in \mathbb{X}_n^d$ and $l \in \mathbb{N}$, such that, for every $(l, x) \in \mathbb{N} \times \mathbb{X}_n^d$, $\tilde{U}_{n,l}^x(t)$ is $\mathscr{F}_t$-adapted, and equation (1.13) holds $\mathbb{P}$-a.s. We call $\tilde{U}_n$ a limit solution to the BTRW SIE (1.11) if $\tilde{U}_n$ is a limit of the truncated solutions $\tilde{U}_{n,l}$ (as $l \to \infty$). If desired, we may indicate the limit type (a.s., in L^p, weak, ..., etc).

Remark 1.2. Of course, in both (1.13) and (1.11), $\tilde{U}_{n,D}^x(t) = \mathbb{E}\left[u_0 \left(\mathbb{S}_{B, \delta_n}^x(t) \right) \right]$. So, by Lemma 1.1, $\tilde{U}_{n,D}^x(t)$ is differentiable in time t and satisfies (1.9). Also, using linear interpolation, we can extend the definition of an already continuous-in-time process $\tilde{U}_n^x(t)$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$, so as to obtain a continuous process on $\mathbb{R}_+ \times \mathbb{R}^d$, for each $n \in \mathbb{N}$, which we will also denote by $\tilde{U}_n^x(t)$. Henceforth, any such sequence $\{\tilde{U}_n\}$ of interpolated $\tilde{U}_n$'s will be called a continuous or an interpolated solution to the system $\left\{ e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n) \right\}_{n=1}^{\infty}$. Similar comments apply to solutions of the truncated $e_{\mathrm{BTRW}}^{\mathrm{t\text{-}SIE}}(a, u_0, n, l)$.

We now define solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ based entirely on their approximating $\left\{e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)\right\}$, through their limit. Since we defined direct and limit solutions to $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$, for each fixed n, we get two types of BTRW SIEs limit solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$: direct BTRW SIEs limit solutions and BTRW SIE double limit solutions. The "double" in the second type of solutions reminds us that we are taking two limits, one from truncated to nontruncated fixed lattice (as $l \to \infty$) and the other limit is taken as the lattice mesh size shrinks to zero (as $\delta_n \searrow 0$ or equivalently as $n \nearrow \infty$).

Definition 1.4 (BTRW SIEs limits solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$). We say that the random field U is a BTRW SIE limit solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ iff there is a solution $\{\tilde{U}_n^x(t)\}_{n \in \mathbb{N}}$ to the lattice SIE system $\left\{e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)\right\}_{n \in \mathbb{N}}$ on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ and with respect to a Brownian system $\{W_n^x(t)\}_{(n,x) \in \mathbb{N} \times \mathbb{X}_n^d}$ such that U is the limit or a modification of the limit of $\left\{\tilde{U}_n\right\}_{n \in \mathbb{N}}$ (or a subsequence thereof). A BTRW SIE limit solution U is called a direct BTRW SIEs limit solution or a BTRW SIEs double limit solution according as $\{\tilde{U}_n^x(t)\}_{n \in \mathbb{N}}$ is a sequence of direct or limit solutions to $\left\{e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)\right\}_{n \in \mathbb{N}}$. The limits may be taken in the a.s., in probability, L^p, weak, or any other suitable sense. We say that uniqueness in law holds if whenever $U^{(1)}$ and $U^{(2)}$ are BTRW SIEs limit solutions they have the same law. We say that pathwise uniqueness holds for BTRW SIEs limit solutions if whenever $\left\{\tilde{U}_n^{(1)}\right\}$ and $\left\{\tilde{U}_n^{(2)}\right\}$ are lattice SIEs solutions on the same probability space and with respect to the same Brownian system, their limits $U^{(1)}$ and $U^{(2)}$ are indistinguishable.

Remark 1.3. When desired, the types of the solution and the limit are explicitly stated (e.g., we say strong (weak) BTRW SIEs weak, in probability, $L^p(\Omega)$, or a.s. limit solution to indicate that the solution to the approximating SIEs system is strong (weak) and that the limit of the SIEs is in the weak, in the probability, in the $L^p(\Omega)$, or in the a.s. sense, respectively).

1.4. **The K-martingale approach.** We now describe our K-martingale approach, which is tailor-made for kernel SIEs like $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ and other mild formulations for many SPDEs on the lattice. The first step is to truncate to a finite lattice model as in (1.13). Of course, even after we truncate the lattice, a remaining hurdle to applying a martingale problem approach is that the finite sum of stochastic integrals in (1.13) is not a local martingale. So, we introduce a key ingredient in this K-martingale method: the auxiliary problem associated with the truncated BTRW SIE in (1.13), which we now give. Fix $(l, n) \in \mathbb{N}^2$ and $\tau \in \mathbb{R}_+$. We define the τ-auxiliary BTRW SIE associated with (1.13) on $[0, \tau] \times \mathbb{X}_n^d$ by

$$
(\text{Aux}) \quad X_{n,l}^{\tau;x}(t) = \begin{cases} \tilde{U}_{n,D}^x(t) + \displaystyle\sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n, s, \tau}^{x,y}\left(X_{n,l}^{\tau,y}(s)\right) dW_n^y(s); & x \in \mathbb{X}_{n,l}^d, \\ \tilde{U}_{n,D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d \end{cases}
$$

where the independent BMs sequence $\{W_n^y\}_{y \in \mathbb{X}_{n,l}^d}$ in (Aux) is the same for all $\tau > 0$, as well as $x \in \mathbb{X}_{n,l}^d$. We denote (Aux) by $e_{\mathrm{BTRW}}^{\mathrm{aux\text{-}SIE}}(a, u_0, n, l, \tau)$. We say that the pair

of families $\left(\left\{ X_{n,l}^{\tau} \right\}_{\tau \geq 0}, \{W_n^y\}_{y \in \mathbb{X}_{n,l}^d} \right)$ is a solution to $\left\{ e_{\mathrm{BTRW}}^{\mathrm{aux\text{-}SIE}}(a, u_0, n, l, \tau) \right\}_{\tau \geq 0}$ on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ if there is one family of independent BMs (up to indistinguishability) $\{W_n^y(t); 0 \leq t < \infty\}_{y \in \mathbb{X}_{n,l}^d}$ on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ such that, for every fixed $\tau \in \mathbb{R}_+$

(a) the process $\left\{ X_{n,l}^{\tau,x}(t), \mathscr{F}_t; 0 \leq t \leq \tau, x \in \mathbb{X}_n^d \right\}$ has continuous sample paths in t for each fixed $x \in \mathbb{X}_n^d$ and $X_{n,l}^{\tau,x}(t) \in \mathscr{F}_t$ for all $x \in \mathbb{X}_n^d$ for every $0 \leq t \leq \tau$; and

(b) equation (Aux) holds on $[0, \tau] \times \mathbb{X}_n^d$, $\mathbb{P}$-almost surely.

Naturally, implicit in our definition above the assumption that, for each fixed $\tau \in \mathbb{R}_+$, we have

$$\mathbb{P}\left[\int_0^t \left(\kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) \right)^2 ds < \infty \right] = 1; \ \forall x, y \in \mathbb{X}_{n,l}^d, 0 \leq t \leq \tau.$$

For simplicity, we will sometimes say that $X_{n,l}^{\tau} = \left\{ X_{n,l}^{\tau,x}(t), \mathscr{F}_t; 0 \leq t \leq \tau, x \in \mathbb{X}_n^d \right\}$ is a solution to (Aux) to mean the above. Clearly, if $X_{n,l}^{\tau,x}(t)$ satisfies (Aux) then $\tilde{U}_{n,l}^x(\tau) := X_{n,l}^{\tau,x}(\tau)$ satisfies (1.13) at $t = \tau$ for all $x \in \mathbb{X}_n^d$. Also, for each n and each $1 \leq d \leq 3$

$$\left| \kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) \right| = \left| \frac{\mathbb{K}_{\tau - s; x, y}^{\mathrm{BTRW}_{\delta_n}^d}}{\delta_n^{d/2}} a(X_{n,l}^{\tau,y}(s)) \right| \leq \frac{\left| a\left(X_{n,l}^{\tau,y}(s) \right) \right|}{\delta_n^{d/2}}.$$

In addition, for each fixed $\tau \in \mathbb{R}_+$ and each fixed $x, y \in \mathbb{X}_{n,l}^d$ we have for a solution $X_{n,l}^{\tau}$ to (Aux) that

$$\kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) \in \mathscr{F}_s; \quad \forall s \leq \tau,$$

since, of course the deterministic $\mathbb{K}_{\tau - s; x, y}^{\mathrm{BTRW}_{\delta_n}^d} / \delta_n^{d/2} \in \mathscr{F}_s$ and $a(X_{n,l}^{\tau,y}(s)) \in \mathscr{F}_s$. Thus, if $X_{n,l}^{\tau}$ solves (Aux); then, for each fixed $\tau > 0$ and $x, y \in \mathbb{X}_{n,l}^d$, each stochastic integral in (Aux)

$$I_{n,l}^{\tau,x,y} = \left\{ I_{n,l}^{\tau,x,y}(t) := \int_0^t \kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) dW_n^y(s), \mathscr{F}_t; \ 0 \leq t \leq \tau \right\}$$

is a continuous local martingale in t on $[0, \tau]$. This is clear since by a standard localization argument we may assume the boundedness of a ($|a(u)| \leq C$); in this case we have for each fixed $x, y \in \mathbb{X}_{n,l}^d$ and $\tau \in \mathbb{R}_+$ that

$$\mathbb{E}\left[I_{n,l}^{\tau,x,y}(t) \middle| \mathscr{F}_r \right] = \int_0^r \kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) dW_n^y(s) = I_{n,l}^{\tau,x,y}(r), \ r \leq t \leq \tau.$$

So, the finite sum over $\mathbb{X}_{n,l}^d$ in (Aux) is also a continuous local martingale in t on $[0, \tau]$. I.e., for each $\tau > 0$ and $x \in \mathbb{X}_{n,l}^d$

$$M_{n,l}^{\tau,x} = \left\{ M_{n,l}^{\tau,x}(t) := \sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n, s, \tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) dW_n^y(s), \mathscr{F}_t; \ 0 \leq t \leq \tau \right\} \in \mathscr{M}_2^{c,loc}$$

with quadratic variation

$$(1.14) \qquad \left\langle M^{\tau,x}_{n,l}(\cdot) \right\rangle_t = \sum_{y \in \mathbb{X}^d_{n,l}} \int_0^t \left[\kappa^{x,y}_{\delta_n,s,\tau} \left(X^{\tau,y}_{n,l}(s) \right) \right]^2 ds$$

where we have used the independence of the BMs $\{W^y_n\}_{y \in \mathbb{X}^d_{n,l}}$ within the lattice $\mathbb{X}^d_{n,l}$. For each $\tau > 0$, we call $M^{x,\tau}_{n,l}$ a kernel local martingale (or K-local martingale).

There is another complicating factor in formulating our K-martingale problem approach that is not present in the standard SDEs setting. To easily extract solutions to the truncated BTRW SIEs in (1.13) from the family of auxiliary problems $\left\{ e^{\text{aux-SIE}}_{\text{BTRW}}(a, u_0, n, l, \tau) \right\}_{\tau > 0}$ in (Aux), we want the independent BMs sequence $\{W^y_n\}_{y \in \mathbb{X}^d_{n,l}}$ to not depend on the choices of τ and x. I.e., we want all the K-local martingales in (Aux) to be stochastic integrals with respect to the same sequence $\{W^y_n\}_{y \in \mathbb{X}^d_{n,l}}$, regardless of τ and x. With this in mind, we now formulate the K-martingale problem associated with the auxiliary BTRW SIEs in (Aux). Let

$$(1.15) \qquad \mathrm{C}_{n,l} := \left\{ u : \mathbb{R}_+ \times \left(\mathbb{X}^d_{n,l} \right)^2 \to \mathbb{R}^2 ; t \mapsto u^{x_1,x_2}(t) \text{ is continuous } \forall x_1, x_2 \right\}.$$

For $u \in \mathrm{C}_{n,l}$, let $u^{x_1,x_2}(t) = (u^{x_1}_1(t), u^{x_2}_2(t))$ with $u^x(t) = u^{x,x}(t)$; and for any $\tau_1, \tau_2 > 0$ and any $x_1, x_2, y \in \mathbb{X}^d_{n,l}$ let

$$(1.16) \qquad \Upsilon^{x_{i,j},y}_{\delta_n,t,\tau_{i,j}} \left(u^y(t) \right) := \frac{\mathbb{K}^{\text{BTRW}^d_{\delta_n}}_{\tau_i - t; x_i, y}}{\delta_n^{d/2}} a(u^y_i(t)) \frac{\mathbb{K}^{\text{BTRW}^d_{\delta_n}}_{\tau_j - t; x_j, y}}{\delta_n^{d/2}} a(u^y_j(t)); \quad 1 \le i, j \le 2,$$

(we are allowing the cases $\tau_1 = \tau_2$ and/or $x_1 = x_2$) where for typesetting convenience we denoted the points (τ_i, τ_j) and (x_i, x_j) by $\tau_{i,j}$ and $x_{i,j}$, respectively. We denote by ∂_i and ∂^2_{ij} the first order partial derivative with respect to the i-th argument and the second order partials with respect to the i and j arguments, respectively. Let $\mathrm{C}^2 = \mathrm{C}^2(\mathbb{R}^2; \mathbb{R})$ be the class of twice differentiable real-valued functions on $\mathbb{R}^2$ and let

$$(1.17) \qquad \mathrm{C}^2_b = \left\{ f \in \mathrm{C}^2 ; f \text{ and all its partial derivatives are bounded} \right\}.$$

Now, for $\tau_1, \tau_2 > 0$, for $f \in \mathrm{C}^2_b$, and for $(t, x_1, x_2, u) \in [0, \tau_1 \wedge \tau_2] \times \left(\mathbb{X}^d_{n,l} \right)^2 \times \mathrm{C}_{n,l}$ let

$$(1.18) \qquad \begin{aligned} \left(\mathscr{A}^{\tau_{1,2}}_\Upsilon f \right) (t, x_1, x_2, u) &:= \sum_{1 \le i \le 2} \partial_i f \left(u^{x_1,x_2}(t) \right) \frac{\partial}{\partial t} \tilde{U}^{x_i}_{n,D}(t) \\ &\quad + \frac{1}{2} \sum_{1 \le i,j \le 2} \partial^2_{ij} f \left(u^{x_i,x_j}(t) \right) \sum_{y \in \mathbb{X}^d_{n,l}} \Upsilon^{x_{i,j},y}_{\delta_n,t,\tau_{i,j}} \left(u^y(t) \right) \end{aligned}$$

Let $X^\tau_{n,l} = \left\{ X^{\tau,x}_{n,l}(t); 0 \le t \le \tau, x \in \mathbb{X}^d_n \right\}$ be a continuous in t adapted real-valued process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$. For every $\tau_1, \tau_2 > 0$ define the two-dimensional stochastic process $Z^{\tau_{1,2}}_{n,l}$:

$$(1.19) \qquad \left\{ Z^{x_{1,2},\tau_{1,2}}_{n,l}(t) = \left(X^{\tau_1,x_1}_{n,l}(t), X^{\tau_2,x_2}_{n,l}(t) \right) ; (t, x_1, x_2) \in [0, \tau_1 \wedge \tau_2] \times \left(\mathbb{X}^d_{n,l} \right)^2 \right\}$$

with $Z_{n,l}^{y,\tau_{1,2}}(t) = \left(X_{n,l}^{\tau_1,y}(t), X_{n,l}^{\tau_2,y}(t) \right)$ and let $U_0^{x_1,x_2} = (u_0(x_1), u_0(x_2))$. We say that the family $\left\{ X_{n,l}^{\tau} \right\}_{\tau \geq 0}$ satisfies the K-martingale problem associated with the auxiliary BTRW SIEs in (Aux) on $\mathbb{R}_+ \times \mathbb{X}_n^d$ if for every $f \in C_b^2$, $0 < \tau_1, \tau_2 < \infty$, $\tau = \tau_1 \wedge \tau_2$, $t \in [0,\tau]$, $x_1, x_2 \in \mathbb{X}_{n,l}^d$, and $x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d$ we have
(KM)
$$\begin{cases} f(Z_{n,l}^{x_{1,2},\tau_{1,2}}(t)) - f(U_0^{x_1,x_2}) - \int_0^t \left(\mathscr{A}_{\Upsilon}^{\tau_{1,2}} f \right)(s, x_1, x_2, Z_{n,l}^{\tau_{1,2}})ds \in \mathscr{M}_2^{c,loc}; \\ X_{n,l}^{\tau,x}(t) = \tilde{U}_{n,D}^x(t). \end{cases}$$

We are now ready to state the equivalence of the K-martingale problem in (KM) to the auxiliary SIEs in (Aux) and its implication for the BTRW SIE in (1.13).

Theorem 1.1. *The existence of a solution pair* $\left(\left\{ X_{n,l}^{\tau} \right\}_{\tau \geq 0}, \{W_n^y\}_{y \in \mathbb{X}_{n,l}^d} \right)$ *to* $\left\{ e_{BTRW}^{aux\text{-}SIE}(a, u_0, n, l, \tau) \right\}_{\tau \geq 0}$ *in* (Aux) *on a filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ *is equivalent to the existence of a family of processes* $\left\{ X_{n,l}^{\tau} \right\}_{\tau \geq 0}$ *satisfying* (KM). *Furthermore, if there is* $\left\{ X_{n,l}^{\tau} \right\}_{\tau \geq 0}$ *satisfying* (KM) *then there is a solution to* (1.13) *on* $\mathbb{R}_+ \times \mathbb{X}_n^d$.

The versatility of the K-martingale approach and the fact that it represents a unified way of dealing with many SPDEs that are of different orders is now clear. The kernel of our BTRW in (Aux) and (KM) may be replaced by the discretized version of the linearized Kuramoto-Sivashinsky kernel (the spatially discretized version of the IBTBAP kernel in (KSK)) or by the density of a random walk to handle in a unified method the fourth order linearized KS (1.22) and other related SPDEs or the second order RD SPDEs. Only minor and obvious modifications are needed to apply this approach to Burgers-type SPDEs. It can also be adapted to treat Navier-Stokes SPDEs driven by space-time white noise and many hyperbolic SPDEs.

1.5. **Main theorem.** We denote by $H^{\gamma_t, \gamma_s}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ the space of real-valued locally Hölder functions on $\mathbb{T} \times \mathbb{R}^d$ whose time and space Hölder exponents are in $(0, \gamma_t)$ and $(0, \gamma_s)$, respectively. The following theorem gives our main limit solutions result for $e_{BTP}^{SIE}(a, u_0)$ under both Lipschitz and non-Lipschitz conditions.

Theorem 1.2 (BTRW SIEs limit solutions to $e_{BTP}^{SIE}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$). *The following existence, uniqueness, and regularity results for the BTP SIEs* (1.3) *hold*

    (a) *Under the Lipschitz conditions*

(Lip) $\begin{cases} (a) & |a(u) - a(v)| \leq K|u - v| \quad u, v \in \mathbb{R}; \\ (b) & a^2(u) \leq C(1 + u^2), \quad u \in \mathbb{R}; \\ (c) & u_0 \in C^2(\mathbb{R}^d; \mathbb{R}) \text{ and nonrandom and bounded with } D_{ij}u_0 \\ & \text{bounded and Hölder continuous } \forall\ 1 \leq i, j \leq d \end{cases}$

    there exists a unique-in-law direct BTRW SIE limit solution to $e_{BTP}^{SIE}(a, u_0)$, *U, such that $U(t, x)$ is $L^p(\Omega, \mathbb{P})$-bounded on $\mathbb{T} \times \mathbb{R}^d$ for every $p \geq 2$ and*

$U \in \mathrm{H}^{\frac{4-d}{8},\,\alpha_d}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ *for every* $1 \le d \le 3$ *and every* $\alpha_d \in I_d$, *where*

$$I_d = \begin{cases} (0,1]; & d = 1, \\ (0,1); & d = 2, \\ (0,\frac{1}{2}); & d = 3. \end{cases}$$

I.e., $U \in \mathrm{H}^{\frac{3}{8},1}(\mathbb{T}\times\mathbb{R};\mathbb{R})$, $U \in \mathrm{H}^{\frac{1}{4},1}(\mathbb{T}\times\mathbb{R}^2;\mathbb{R})$, *and* $U \in \mathrm{H}^{\frac{1}{8},\frac{1}{2}}(\mathbb{T}\times\mathbb{R}^3;\mathbb{R})$.

(b) *Assume the less-than-Lipschitz conditions* (NLip) *hold. Then, there exists a BTRW SIE double limit solution to* $e_{BTP}^{SIE}(a,u_0)$, U, *such that* $U(t,x)$ *is* $L^p(\Omega,\mathbb{P})$-*bounded on* $\mathbb{T} \times \mathbb{R}^d$ *for every* $p \ge 2$ *and* $U \in \mathrm{H}^{\frac{4-d}{8},\,\alpha_d}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ *for every* $1 \le d \le 3$ *and every* $\alpha_d \in I_d$ *as in part* (a).

Remark 1.4. There is a subtle distinction between the spatial regularity in the one and two dimensional cases. We explore this further in [5] (see also Remark 2.2). In Appendix B we prove existence, pathwise uniqueness, and $L^p(\Omega)$-boundedness for our BTP SIE (1.3) and the same local Hölder regularity as in Theorem 1.2 directly for $1 \le d \le 3$ under Lipschitz conditions, without discretization. With extra work, it is possible to prove the existence of a strong limit solution under Lipschitz conditions. We plan to address that in a future article. Of course, we can use change of measure—as we did in our earlier work on Allen-Cahn SPDEs and other second order SPDEs (see e.g. [10] and all our change of measure references in [7] for results and conditions)—to transfer existence, uniqueness, and law equivalence results between $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ and the BTP SIE with measurable drift $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,b,u_0)$:

$$
\begin{aligned}
(1.20) \qquad U(t,x) = & \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y)dy + \int_{\mathbb{R}^d}\int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} b(U(s,y))dsdy \\
& + \int_{\mathbb{R}^d}\int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} a(U(s,y))\mathscr{W}(ds \times dy),
\end{aligned}
$$

under the same conditions on the drift/diffusion ratio. If it is desired to investigate $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,b,u_0)$ on a bounded domain in $\mathbb{R}^d$ with a regular boundary, we simply replace the BTBM density $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ in (1.20) with its boundary-reflected or boundary-absorbed version (the BTBM density in which the outside d-dimensional BM is either reflected or absorbed at the boundary).

1.6. **BTP SIEs and their fourth order SPDEs links.** In this section, we briefly discuss the quite interesting connections—both direct and indirect—of the BTP SIEs $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ to fourth order SPDEs driven by space-time white noise. We start by giving a quick glimpse into the indirect link of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ to Kuramoto-Sivashinsky type SPDEs via our imaginary-Brownian-time-Brownian-angle process (IBTBAP) representation of the linearized KS PDE (see [2, 1]). A more extensive treatment of such SPDEs using this IBTBAP approach is presented in [3, 4, 6]. We then end this subsection by giving a connection of the BTP SIEs $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ to an unconventional new parametrized fourth order BTP SPDEs using the spatially discretized versions of these equations.

1.6.1. *BTP-SIEs are cousins of the Kuramoto-Sivashinsky and related SPDEs.* By replacing $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ and $\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d}$ in (1.3) with the intimately connected kernels $\mathbb{K}_{t;x,y}^{\mathrm{LKS}^d}$

and $\mathbb{K}^{\mathrm{LKS}^d}_{t-s;x,y}$, defined by

(KSK)
$$\begin{cases} K^{\mathrm{CLKS}^d}_{is;x,y} := \dfrac{\exp(is)}{(2\pi i s)^{d/2}} e^{-|x-y|^2/2is}, \\[2mm] \mathbb{K}^{\mathrm{LKS}^d}_{t;x,y} := \displaystyle\int_{-\infty}^{0} K^{\mathrm{CLKS}^d}_{is;x,y} K^{\mathrm{BM}}_{t;0,s} ds + \int_{0}^{\infty} K^{\mathrm{CLKS}^d}_{is;x,y} K^{\mathrm{BM}}_{t;0,s} ds \end{cases}$$

we obtain our definition of IBTBAP solutions

$$(1.21) \qquad U(t,x) = \int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{LKS}^d}_{t;x,y} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}^{\mathrm{LKS}^d}_{t-s;x,y} a(U(s,y)) \mathscr{W}(ds \times dy)$$

to the canonical Kuramoto-Sivashinsky SPDE with linearized PDE part:

$$(1.22) \qquad \begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{1}{8}\Delta^2 U - \dfrac{1}{2}\Delta U - \dfrac{1}{2}U + a(U)\dfrac{\partial^2 W}{\partial t \partial x}, & (t,x) \in (0,+\infty) \times \mathbb{R}^d; \\[2mm] U(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

This IBTBAP representation approach for the SPDE (1.22) is inspired by our earlier work [2], in which we used the deterministic version of (1.21) ($a \equiv 0$) to solve the linearized KS PDE obtained from (1.22) by setting $a \equiv 0$. It easily allows for the addition of a nonlinear $f(u)$ to (1.22). The nonlinearity $f(u)$ could be an Allen-Cahn one (to get Swift-Hohenberg SPDE), KPP (a new KS-type SPDE), Burgers (versions and variants of KS SPDE), and many more interesting nonlinearities. Quantum mechanics experts will immediately note that, except for the $\exp(is)$ term, $K^{\mathrm{CLKS}^d}_{is;x,y}$ in the definition of the IBTBAP kernel in (KSK) is a d-dimensional version of the free propagator associated with Schrödinger equation.

One reason why the BTP-SIEs are cousins of the Kuramoto-Sivashinsky and related SPDEs is that the kernel $\mathbb{K}^{\mathrm{LKS}^d}_{t;x,y}$ above may be regarded as the "density" of the IBTBAP, as in [2]. By construction, the IBTBAP—which we also called the linearized Kuramoto-Sivashinsky process or LKSP—is intimately connected to BTPs (e.g. [9, 8, 2]). Moreover, as we showed in [9, 8, 2] for the PDEs case, the kernels $\mathbb{K}^{\mathrm{BTBM}^d}_{t;x,y}$ and $\mathbb{K}^{\mathrm{LKS}^d}_{t;x,y}$ have similar regularizing effects on their corresponding equations. Analogously, our solutions to the BTP SIEs (1.3) have similar d-dependent regularity properties to the IBTBAP mild solutions (1.21) to KS-type SPDEs, including (1.22) and nonlinear versions of it, for $d = 1, 2, 3$ (see [3] and followup papers). We are also currently using this approach to investigate asymptotic and other qualitative behaviors of several related nonlinear SPDEs in applied mathematics (e.g., [4, 6]).

1.6.2. *BTP SIEs and the parametrized BTP SPDEs.* We now link $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ to a new fourth order parametrized SPDE on discrete spatial lattices. First, we note that this direct link of BTP SIEs to SPDEs is not as straightforward as one might be tempted to believe; and it cannot be resolved by using a conventional SPDE, as we explain below. Even though it is certainly true that the deterministic term on the right hand side of (1.3)

$$(1.23) \qquad v(t,x) := \int_{\mathbb{R}^d} \mathbb{K}^{\mathrm{BTBM}^d}_{t;x,y} u_0(y) dy$$

solves the deterministic fourth order PDE (1.1) (see Allouba et al. [8, 9]), it turns out that the BTP SIE $e^{\mathrm{SIE}}_{\mathrm{BTP}}(a, u_0)$ is *different* from—i.e., not the mild stochastic integral form of—the naturally guessed SPDE (1.5) (see [1]). This is due precisely to the Laplacian acting on the initial data u_0. Instead, (1.5) may be viewed as a

degenerate version of the parametrized BTP SPDEs linked here to our BTP SIEs (see Appendix C for a brief discussion involving the kernel formulation of (1.5) in terms of its spatially-discretized version). So, calling $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ the BTP solution to (1.5)—as we did in [1]—is not very precise, and we call $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ the BTP SIE instead. Our main interest is in the BTP SIEs, and here also we use the spatially-discretized version of our BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ (the BTRW SIE $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ in (1.11)) to connect it to fourth order SPDEs. Figuring out this correct and subtle SPDE-link to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ via our easier-to-see discretized versions of the equations is another advantage of this multiscale approach over the direct one.

To intuitively see the correct SPDE link to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$, we go back to the deterministic BTP PDE case (1.1), and observe again that the solution in (1.23) $v \in \mathrm{C}^{1,4}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})$ is indeed very smooth for *all* times and *all* spatial dimensions $d \geq 1$ (see [8, 9]) despite the presence of the positive biLaplacian. In order to heuristically pinpoint the cause of the smoothing effect of the BTBM kernel $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ in (1.23) and connect it to the corresponding PDE (1.1), we must look at both terms $\Delta^2 u$ and $\Delta u_0/\sqrt{8\pi t}$ *together*. We observe that the smoothing effect of the Laplacian term in (1.1) gets arbitrarily large as the time t in $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ goes to zero ($t \searrow 0$) and fades as $t \nearrow \infty$ at the rate of $1/\sqrt{8\pi t}$; and the Laplacian is acting on the smooth initial solution u_0 (the solution at time $t = 0$). Heuristically, this suggests a recipe for obtaining eternally-smooth solutions involving the positive biLaplacian coupled with the smoothing Laplacian.

Turning our attention now towards the BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. We first give a heuristic prelude to the new notion of parametrized BTP SPDEs, and then we precisely state the result. We know from the results of Sections 1 and 2, (as well as those in Appendix B and the comments in Section 3) that the BTBM density $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ has a significant smoothing effect on solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ as compared to the Brownian motion density in the mild formulation of standard second order RD SPDEs. We also see that the stochastic white noise term in $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ involves the BTBM density at $t - s$, viz. $\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d}$. So, based on the heuristic above and generalizing it, any SPDE that captures the $\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d}$ smoothing effect will include a positive bi-Laplacian term along with a Laplacian term whose coefficient grows arbitrarily large as $s \nearrow t$ at the rate of $1/\sqrt{8\pi(t - s)}$. On one hand, this Laplacian will have to act on the solution of the SPDE for all times $r \leq s$; on the other hand it has to also act on the solution at spatial points $x = y$, since $\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} = 0$ except at $x = y$ when $t - s = 0$. That is, we need to keep track of four parameters in both time and space (not just (t, x)) to encode the smoothing effect of the BTBM kernel in (1.3) into an SPDE; and we are led to the notion of parametrized SPDEs associated with our $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. We give this link on the lattice in Lemma 1.2, and we prove it in Section 2.5 for spatially discretized versions of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. More precisely, we discretize $\mathbb{R}^d$ into the lattices $\mathbb{X}_n^d$. We then fix an arbitrary $n \in \mathbb{N}$ and consider the following parametrized stochastic differential-difference equation

(PSDDE), written in integral form as:

$$
\begin{aligned}
(1.24) \qquad \tilde{U}_n^{x,y}(s,t) - u_0(y) &= \int_0^s \left[\frac{\Delta_n \tilde{U}_n^{x,y}(r,t)\big|_{y=x}}{\sqrt{8\pi(t-r)}} + \frac{1}{8}\Delta_n^2 \tilde{U}_n^{x,y}(r,t) \right] dr \\
&+ \int_0^s a(\tilde{U}_n^y(r)) \frac{dW_n^y(r)}{\delta_n^{d/2}}; \ 0 \le s \le t < \infty, \ x,y \in \mathbb{X}_n^d,
\end{aligned}
$$

which we denote by $e_{\mathrm{BTRW}}^{\mathrm{PSDDE}}(a, u_0, n)$, where Δ_n and Δ_n^2 are the d-dimensional discrete Laplacian and bi-Laplacian on $\mathbb{X}_n^d$, acting on the second spatial argument respectively:

$$
\begin{aligned}
\Delta_n \tilde{U}_n^{x,y}(r,t)\big|_{y=x} &= \sum_{i=1}^d \Delta_{n,i} \tilde{U}_n^{x,y}(r,t)\big|_{y=x} \\
(1.25) \qquad &:= \sum_{i=1}^d \frac{\tilde{U}_n^{x;x_1,\ldots,x_i+\delta_n,\ldots,x_d}(r,t) - 2\tilde{U}_n^{x,x}(r,t) + \tilde{U}_n^{x;x_1,\ldots,x_i-\delta_n,\ldots,x_d}(r,t)}{\delta_n^2}; \\
\Delta_n^2 \tilde{U}_n^{x,y}(r,t) &= \sum_{i,j=1}^d \Delta_{n,j}\Delta_{n,i} \tilde{U}_n^{x,y}(r,t)
\end{aligned}
$$

and where $\tilde{U}_n^y(r) := \tilde{U}_n^{y,y}(r,r)$, and $\tilde{U}_n^{x,y}(0,t) = u_0(y)$ for all $(t,x,y) \in \mathbb{R}_+ \times \mathbb{X}_n^{2d}$. By a solution to the PSDDE system $\left\{ e_{\mathrm{BTRW}}^{\mathrm{PSDDE}}(a, u_0, n) \right\}_{n=1}^\infty$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$ with respect to the Brownian (in t) system $\{W_n^y(t)\}_{(n,y) \in \mathbb{N} \times \mathbb{X}_n^d}$ on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ we mean a sequence of real-valued processes $\left\{ \tilde{U}_n \right\}_{n=1}^\infty$ with continuous sample paths in s for each fixed $(t,x,y) \in \mathbb{R}_+ \times \mathbb{X}_n^{2d}$ and $n \in \mathbb{N}$ such that, for every $(n,t,x,y) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{X}_n^{2d}$, $\tilde{U}_n^{x,y}(s,t)$ is $\mathscr{F}_s$-adapted, and equation (1.24) holds $\mathbb{P}$-a.s. In particular,

$$
\mathbb{P}\left[\int_0^s \left(\left| \frac{\Delta_n \tilde{U}_n^{x,y}(r,t)\big|_{y=x}}{\sqrt{8\pi(t-r)}} + \frac{1}{8}\Delta_n^2 \tilde{U}_n^{x,y}(r,t) \right| + \frac{a^2(\tilde{U}_n^y(r))}{\delta_n^d} \right) dr < \infty \right] = 1
$$

holds for every $0 \le s \le t < \infty$, $x,y \in \mathbb{X}_n^d$.

We then prove (see Section 2.5) that if $\tilde{U}_n^{x,y}(s,t)$ solves (1.24) then $\tilde{U}_n^x(t)$ solves the spatially-discretized version of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$—the BTRW SIE $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ given by (1.11). We stress here that it is not enough for the "diagonal terms" $\tilde{U}_n^{x,x}(t,t) = \tilde{U}_n^x(t)$ to satisfy (1.24) (the special case of (1.24) $s = t$ and $y = x$) to conclude that $\tilde{U}_n^x(t)$ satisfies the BTRW SIE in (1.11): *all* of the $\tilde{U}_n^{x,y}(s,t)$ must satisfy the PSDDE (1.24) for us to have this implication (see Lemma 1.2 and its proof in Section 2.5).

It is in the above spatially-discretized sense that we say that the BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ is associated with the parametrized BTP SPDE

$$
\begin{aligned}
(1.26) \qquad \frac{\partial U_t^x(s,y)}{\partial s} &= \left[\frac{\Delta U_t^x(s,y)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \frac{1}{8}\Delta^2 U_t^x(s,y) \right] ds + a(U_s^y(s,y)) \frac{\partial^2 W}{\partial s \partial y} \\
U_t^x(0,y) &= u_0(y)
\end{aligned}
$$

The system of parametrized stochastic differential-difference equations (PSD-DEs) (1.24), may also be written in differential form as:
(1.27)
$$\begin{cases} d\tilde{U}_n^{x,y}(s,t) = \left[\dfrac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \dfrac{1}{8}\Delta_n^2 \tilde{U}_n^{x,y}(s,t)\right] ds + a(\tilde{U}_n^y(s))\dfrac{dW_n^y(s)}{\delta_n^{d/2}}; \\ \tilde{U}_n^{x,y}(0,t) = u_0(y); \end{cases}$$

The proof of the following lemma follows from an application of Itô's rule. The interesting point here is the unorthodox nature of the parametrized SDDE connected to the lattice version of our BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$.

Lemma 1.2 (Relation between $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a,u_0,n)$ and $e_{\mathrm{BTRW}}^{\mathrm{PSDDE}}(a,u_0,n)$). *Fix an arbitrary $n \in \mathbb{N}$ and assume the conditions in* (NLip) *and let*
$$\tilde{U}_n := \left\{\tilde{U}_n^{x,y}(s,t); (s,t,x,y) \in [0,t] \times \mathbb{R}_+ \times \mathbb{X}_n^{2d}\right\}$$
be a continuous-in-s solution to the parametrized SDDEs (1.24) *such that, for any fixed pair (t,x), $\mathbb{E}\left|\tilde{U}_n^{x,y}(s,t)\right|^2 \leq C$ for all $(s,y) \in [0,t] \times \mathbb{X}_n^d$ for some constant $C > 0$. Then $\tilde{U}_n^x(t) := \tilde{U}_n^{x,x}(t,t)$ solves the BTRW SIE* (1.11).

Remark 1.5. The moment boundedness condition in Lemma 1.2 is technical and may be relaxed. Also, in light of the fact that Lemma 1.2 holds for all n, we may call the BTRW SIE $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a,u_0,n)$ the lattice-kernel-diagonal form of the PSDDE (1.24); and we may then call the BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ the limiting-lattice-kernel-diagonal form of the parametrized BTP SPDE (1.26). A converse of Lemma 1.2 is given in Appendix C in Lemma C.1.

2. Proof of results

2.1. **Some BTRW density estimates.** The first set of estimates we need are bounds on the the square of the Brownian-time random walk density $\mathbb{K}_{t,x}^{\mathrm{BTRW}_{\delta_n}^d}$ and its temporal and spatial differences. The method of proof is to reduce, via an asymptotic argument, these estimates to the corresponding ones for the BTBM density $\mathbb{K}_{t;x}^{\mathrm{BTBM}^d}$ and perform the computations in the setting of the BTBM. Since all the results in this part hold for all $n \geq N^*$ (equivalently for all $\delta_n \leq \delta_{N^*}$) for some positive integer N^*, we will suppress the dependence on n, except when it is needed or helpful, to simplify the notation. Also, whenever we need these estimates, we assume that $n \geq N^*$ without explicitly stating it every time; and when we do, we let

(2.1) $\mathbb{N}^* := \{n \in \mathbb{N}; n \geq N^*\}$

We start by observing that in the classical setting of Brownian motion and its discretized version continuous-time random walk on $\mathbb{X}_n^d = \delta_n \mathbb{Z}^d$, we have the following well known asymptotic result relating their densities (see e.g., [33])

(2.2) $K_{t;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{RW}_{\delta_n}^d} \sim K_{t;x,y}^{\mathrm{BM}^d}\delta_n^d$ as $n \to \infty$ (as $\delta_n \to 0$); $\forall t > 0,\ x,y \in \mathbb{R}^d$,

where for each $x \in \mathbb{R}^d$ we use $[x]_{\delta_n}$ to denote the element of $\mathbb{X}_n^d$ obtained by replacing each coordinate x_i with δ_n times the integer part of $\delta_n^{-1}x_i$, and $a_n \sim b_n$

as $n \to \infty$ means $a_n/b_n \to 1$ as $n \to \infty$. Now, for every continuous and bounded $u_0 : \mathbb{R}^d \to \mathbb{R}$, we have

$$(2.3) \quad \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y) \delta_n^d = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y) dy; \ t > 0, \ x \in \mathbb{R}^d, \ d \geq 1,$$

and by the dominated covergence theorem we obtain

$$
\begin{aligned}
(2.4) \quad & \lim_{\delta_n \searrow 0} \left| \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) - \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y) \delta_n^d \right| \\
& = \left| \int_0^\infty \left\{ \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} \left[K_{s;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{RW}_{\delta_n}^d} - K_{s;x,y}^{\mathrm{BM}^d} \delta_n^d \right] u_0(y) \right\} 2 K_{t;0,s}^{\mathrm{BM}} ds \right| = 0
\end{aligned}
$$

for $t > 0$, $x \in \mathbb{R}^d$, and $d \geq 1$; since, by (2.2),

$$\lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} K_{s;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{RW}_{\delta_n}^d} u_0(y) = \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} K_{s;x,y}^{\mathrm{BM}^d} u_0(y) \delta_n^d = \int_{\mathbb{R}^d} K_{s;x,y}^{\mathrm{BM}^d} u_0(y) dy$$

for every $(s,x) \in (0,\infty) \times \mathbb{R}^d$. We then straightforwardly get the following result.

Lemma 2.1 (BTRW density $\sim$ BTP density)**.** *For every continuous and bounded $u_0 : \mathbb{R}^d \to \mathbb{R}$ and for every $d \geq 1$*

$$(2.5) \quad \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} u_0(y) dy; \forall (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

and the following asymptotic relation holds between the BTP and BTRW densities:

$$(2.6) \quad \mathbb{K}_{t;[x]_{\delta_n},[y]_{\delta_n}}^{\mathrm{BTRW}_{\delta_n}^d} \sim \mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d} \delta_n^d \text{ as } n \to \infty \ (as \ \delta_n \to 0); \ t > 0, \ x,y \in \mathbb{R}^d, \ x \neq y.$$

Remark 2.1. Equation (2.5) confirms the intuitively clear fact that the kernel form of the BTRW DDE (1.9) converges pointwise—as $\delta_n \searrow 0$—to the kernel form of its continuous version, the BTP PDE (1.1). We also remind the reader that the right hand side of (2.5) is in $\mathrm{C}^{1,4}$ for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$ under the conditions (NLip) on u_0.

By reducing the computations to the setting of the Brownian-time Brownian motion, using Lemma 2.1 together with scaling, our method of proof for the next three lemmas shows that the estimates in these lemmas all hold for the BTBM density as well as for the BTRW one with obvious changes from the discrete to the continuous settings (see Lemma 2.2, Lemma 2.3, and Lemma 2.4 below). Thus, these lemmas are stated for both densities $\mathbb{K}_{t;x}^{\mathrm{BTRW}_\delta^d}$ and $\mathbb{K}_{t;x}^{\mathrm{BTBM}^d}$. This, in turn, allows us to easily indicate how to prove the same Hölder regularity of solutions to the BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a,u_0)$ directly (without discretization) in the case of Lipschitz conditions (see Appendix B for the detailed existence, uniqueness, and $L^p(\Omega)$-boundedness proof and see Remark 2.3 for the Hölder regularity proof in that case). We start with

Lemma 2.2. *There are constants C and $\tilde{C}$, depending only on d, and a $\delta^* > 0$ such that for all $\delta \leq \delta^*$*

$$\int_{\mathbb{R}^d} \left[\mathbb{K}_{t;x}^{\mathrm{BTBM}^d} \right]^2 dx = \frac{C}{t^{d/4}} \ and \ \sum_{x \in \mathbb{X}^d} \left[\mathbb{K}_{t;x}^{\mathrm{BTRW}_\delta^d} \right]^2 \leq \tilde{C} \frac{\delta^d}{t^{d/4}}; \quad t > 0, \ 1 \leq d \leq 3$$

and hence

$$\int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTBM}^d_\delta}\right]^2 dxds = Ct^{\frac{4-d}{4}} \ \ and \ \int_0^t \sum_{x\in\mathbb{X}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTRW}^d_\delta}\right]^2 ds \le \tilde{C}\delta^d t^{\frac{4-d}{4}};$$

for all $t > 0$, $1 \le d \le 3$.

Proof.   Using Lemma 2.1 we obtain

$$
\begin{aligned}
(2.7) \qquad \lim_{\delta \searrow 0} \sum_{x\in\mathbb{X}^d} \frac{\left[\mathbb{K}_{t;x}^{\mathrm{BTRW}^d_\delta}\right]^2}{\delta^d} &= \int_{\mathbb{R}^d} \left[\mathbb{K}_{t;x}^{\mathrm{BTBM}^d}\right]^2 dx \\
&= 4\left[\int_0^\infty \int_0^\infty \left[\int_{\mathbb{R}^d} K_{s_1;x}^{\mathrm{BM}^d} K_{s_2;x}^{\mathrm{BM}^d} dx\right] K_{t;0,s_1}^{\mathrm{BM}} K_{t;0,s_2}^{\mathrm{BM}} ds_1 ds_2\right] \\
&= 4\left[\int_0^\infty \int_0^\infty \left[\frac{1}{[2\pi(s_1+s_2)]^{d/2}}\right] K_{t;0,s_1}^{\mathrm{BM}} K_{t;0,s_2}^{\mathrm{BM}} ds_1 ds_2\right] \\
&= 4\left[\int_0^{\pi/2} \int_0^\infty \frac{e^{-\rho^2/2t}}{(2\pi)^{\frac{d+2}{2}} [\rho(\sin(\theta)+\cos(\theta))]^{d/2} t} \rho d\rho d\theta\right] \\
&= \begin{cases} \dfrac{4C_d}{t^{d/4}}; & 1 \le d \le 3, \\ \infty; & d \ge 4, \end{cases}
\end{aligned}
$$

with $C_1 \approx 0.0914$, $C_2 \approx 0.0396$, and $C_3 \approx 0.0243$. Then there is a $\delta^* > 0$ such that, whenever $\delta \le \delta^*$, we obtain

$$\frac{1}{\delta^d} \sum_{x\in\mathbb{X}^d} \left[\mathbb{K}_{t;x}^{\mathrm{BTRW}^d_\delta}\right]^2 \le \frac{\tilde{C}_d}{t^{d/4}}; \ 1 \le d \le 3,$$

with a constant $\tilde{C}_d > 4C_d$. The last assertion of the lemma trivially follows upon integration over the time interval $(0, t]$. $\square$

The following lemma is key to our Hölder regularity result in time for $1 \le d \le 3$. We give a probabilistically-flavored proof using the notion of 2-Brownian-times random walk and 2-Brownian-times Brownian motion given below.

Lemma 2.3. *There is a constant C, depending only on d, and a $\delta^* > 0$ such that for $\delta \le \delta^*$*

$$
(2.8) \qquad \begin{cases} \displaystyle\int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{t-s;x}^{\mathrm{BTBM}^d} - \mathbb{K}_{r-s;x}^{\mathrm{BTBM}^d}\right]^2 dxds \le C(t-r)^{\frac{4-d}{4}}; \ 0 < r < t, 1 \le d \le 3, \\ \displaystyle\int_0^t \sum_{x\in\mathbb{X}^d} \left[\mathbb{K}_{t-s;x}^{\mathrm{BTRW}^d_\delta} - \mathbb{K}_{r-s;x}^{\mathrm{BTRW}^d_\delta}\right]^2 ds \le C\delta^d(t-r)^{\frac{4-d}{4}}; \ 0 < r < t, 1 \le d \le 3, \end{cases}
$$

with the convention that $\mathbb{K}_{t;x}^{\mathrm{BTRW}^d_\delta} = 0 = \mathbb{K}_{t;x}^{\mathrm{BTBM}^d}$ if $t < 0$.

Proof. We will prove that

$$(2.9) \qquad \int_0^t \sum_{x \in \mathbb{X}^d} \left[\mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s+(t-r);x} - \mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s;x} \right]^2 ds \leq C\delta^d (t-r)^{\frac{4-d}{4}}; \quad 1 \leq d \leq 3.$$

for all $\delta \leq \delta^*$, for some $\delta^* > 0$, simultaneously with its corresponding BTBM density statement. The first step is to show the identity

$$(2.10) \qquad \sum_{x \in \mathbb{X}^d} \left[\mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s+(t-r);x} - \mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s;x} \right]^2 = \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s+(t-r),s+(t-r);0} + \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s,s;0} - 2\mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s+(t-r),s;0}$$

where

$$(2.11) \qquad \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{u,v;0} = 4 \int_0^\infty \int_0^\infty K^{\mathrm{RW}_\delta^d}_{r_1+r_2;0} K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{v;0,r_2} dr_1 dr_2$$

is the density of the 2-Brownian-times random walk

$$(2.12) \qquad \mathbb{S}^0_{B^{(1)},B^{(2)},\delta_n}(u,v) := S^0_{\delta_n}\left(\left| B^{(1)}_u \right| + \left| B^{(2)}_v \right| \right); \quad 0 \leq u,v < \infty,$$

in which the d-dimensional random walk $S^0_{\delta_n}$ and the two one-dimensional BMs B^1 and B^2 are all independent. But,

$$(2.13) \qquad \begin{aligned} &\sum_{x \in \mathbb{X}^d} \mathbb{K}^{\mathrm{BTRW}_\delta^d}_{u;x} \mathbb{K}^{\mathrm{BTRW}_\delta^d}_{v;x} \\ &= 4 \int_0^\infty \int_0^\infty \left[\sum_{x \in \mathbb{X}^d} K^{\mathrm{RW}_\delta^d}_{r_1;x} K^{\mathrm{RW}_\delta^d}_{r_2;x} \right] K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{v;0,r_2} dr_1 dr_2 \\ &= 4 \int_0^\infty \int_0^\infty K^{\mathrm{RW}_\delta^d}_{r_1+r_2;0} K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{v;0,r_2} dr_1 dr_2 = \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{u,v;0}. \end{aligned}$$

The identity (2.10) immediately follows from (2.13). Similarly, we get the corresponding identity for the BTBM setting

$$(2.14) \qquad \int_{\mathbb{R}^d} \left[\mathbb{K}^{\mathrm{BTBM}_\delta^d}_{s+(t-r);x} - \mathbb{K}^{\mathrm{BTBM}_\delta^d}_{s;x} \right]^2 dx = \mathbb{K}^{\mathrm{BTBM}^d(2)}_{s+(t-r),s+(t-r);0} + \mathbb{K}^{\mathrm{BTBM}^d(2)}_{s,s;0} - 2\mathbb{K}^{\mathrm{BTBM}^d(2)}_{s+(t-r),s;0}$$

where

$$(2.15) \qquad \mathbb{K}^{\mathrm{BTBM}^d(2)}_{u,v;0} = 4 \int_0^\infty \int_0^\infty K^{\mathrm{BM}_\delta^d}_{r_1+r_2;0} K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{v;0,r_2} dr_1 dr_2$$

is the density of the 2-Brownian-times Brownian motion

$$(2.16) \qquad \mathbb{X}^0_{B^{(1)},B^{(2)}}(u,v) := X^0 \left(\left| B^{(1)}_u \right| + \left| B^{(2)}_v \right| \right); \quad 0 \leq u,v < \infty,$$

in which the d-dimensional BM X^0 and the two one dimensional BMs $B^{(1)}$ and $B^{(2)}$ are all independent. Using the identities (2.10) and (2.14), along with a similar asymptotic argument to the one we used in the proof of Lemma 2.2 together with

the dominated convergence theorem, yield

$$
\lim_{\delta \searrow 0} \frac{1}{\delta^d} \left[\int_0^t \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s+(t-r),s+(t-r);0} ds + \int_0^t \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s,s;0} ds - 2 \int_0^t \mathbb{K}^{\mathrm{BTRW}_\delta^d(2)}_{s+(t-r),s;0} ds \right]
$$

$$
= \lim_{\delta \searrow 0} \int_0^t \sum_{x \in \mathbb{X}^d} \frac{\left[\mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s+(t-r);x} - \mathbb{K}^{\mathrm{BTRW}_\delta^d}_{s;x} \right]^2}{\delta^d} ds
$$

$$
= \int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}^{\mathrm{BTBM}_\delta^d}_{s+(t-r);x} - \mathbb{K}^{\mathrm{BTBM}_\delta^d}_{s;x} \right]^2 dx \, ds
$$

(2.17)

$$
= \left[\int_0^t \mathbb{K}^{\mathrm{BTBM}^d(2)}_{s+(t-r),s+(t-r);0} ds + \int_0^t \mathbb{K}^{\mathrm{BTBM}^d(2)}_{s,s;0} ds - 2 \int_0^t \mathbb{K}^{\mathrm{BTBM}^d(2)}_{s+(t-r),s;0} ds \right]
$$

$$
= \left[\int_0^t \tilde{\mathbb{K}}_{2s+2(t-r)} ds + \int_0^t \tilde{\mathbb{K}}_{2s} ds - 2 \int_0^t \tilde{\mathbb{K}}_{2s+(t-r)} ds \right]
$$

$$
= \left[\int_0^{\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s} ds - \int_{\frac{t-r}{2}}^{t-r} \tilde{\mathbb{K}}_{2s} ds - \int_t^{t+\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s} ds + \int_{t+\frac{t-r}{2}}^{2t-r} \tilde{\mathbb{K}}_{2s} ds \right]
$$

for $1 \le d \le 3$, where $\tilde{\mathbb{K}}_w$ is defined in terms of $\mathbb{K}^{\mathrm{BTBM}^d(2)}_{u,v;0}$ by the relation

(2.18) $\quad \tilde{\mathbb{K}}_w = \mathbb{K}^{\mathrm{BTBM}^d(2)}_{u,v;0} \iff w = u + v$ and (u,v) has one of the forms
$(u,v) = (a,a)$ or $(u,v) = (a+b,a)$ or $(u,v) = (a,a+b)$ for some $a, b \ge 0$.

We observe that

$$
\tilde{\mathbb{K}}_{2u} = \mathbb{K}^{\mathrm{BTBM}^d(2)}_{u,u;0} = 4 \int_0^\infty \int_0^\infty K^{\mathrm{BM}_\delta^d}_{r_1+r_2;0} K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{u;0,r_2} dr_1 dr_2
$$

(2.19)

$$
= 4 \int_0^\infty \int_0^\infty \left[\int_{\mathbb{R}^d} K^{\mathrm{BM}_\delta^d}_{r_1;x} K^{\mathrm{BM}_\delta^d}_{r_2;x} dx \right] K^{\mathrm{BM}}_{u;0,r_1} K^{\mathrm{BM}}_{u;0,r_2} dr_1 dr_2
$$

$$
= \int_{\mathbb{R}^d} \left[\mathbb{K}^{\mathrm{BTBM}_\delta^d}_{u;x} \right]^2 ds = \frac{C_d}{u^{d/4}}; \quad 1 \le d \le 3
$$

The last assertion follows from the computation in (2.7) (or see p. 531 in [1]). It is clear then that $\tilde{\mathbb{K}}_{2u}$ is decreasing in u. Thus, the sum of the last three terms of the (2.17) is ≤ 0. This and (2.19) give us (2.9) for all $\delta \le \delta^*$, for some $\delta^* > 0$ and for some constant $C > 0$, together with its corresponding BTBM density statement; and Lemma 2.3 follows at once. $\quad \square$

The following spatial difference second moment inequality for the BTRW and BTBM densities captures their impressive spatial-regularizing effect on our solutions.

Lemma 2.4. For $1 \le d \le 3$, there are intervals

$$
I_d = \begin{cases} (0,1]; & d = 1, \\ (0,1); & d = 2, \\ (0,\frac{1}{2}); & d = 3, \end{cases}
$$

positive numbers $\{\alpha_d \in I_d\}_{d=1}^3$, *constants* $\{C_d\}_{d=1}^3$ *depending only on* d *and* α_d, *and a* $\delta^* > 0$ *such that for* $\delta \leq \delta^*$

$$(2.20) \quad \begin{cases} \int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTBM}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTBM}_\delta^d}\right]^2 dxds \leq C_d |z|^{2\alpha_d} t^{p_d(\alpha_d)}; \quad \forall \alpha_d \in I_d, t > 0, \\ \int_0^t \sum_{x \in \mathbb{X}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTRW}_\delta^d}\right]^2 ds \leq C_d \delta^d |z|^{2\alpha_d} t^{p_d(\alpha_d)}; \quad \forall \alpha_d \in I_d, t > 0, \end{cases}$$

where $0 < C_d < \infty$ *and* $0 \leq p_d(\alpha_d) < 1$ *for every* $\alpha_d \in I_d$ *for* $d = 1, 2, 3$.

Remark 2.2. In the case $d = \alpha_d = 1$ the power $p_1(\alpha_1) = 1/4$. Also, the constants C_d's are increasing in α_d, with $0 < C_1 \leq c < \infty$ for some absolute constant c for all $0 < \alpha_1 \leq 1$; whereas $\lim_{\alpha_2 \to 1} C_2 = +\infty = \lim_{\alpha_3 \to 1/2} C_3$. Moreover, while their exact values are not needed, the following limits hold for the powers $p_d(\alpha_d)$

$$(2.21) \quad \begin{cases} \lim_{\alpha_d \to 1} p_d(\alpha_d) = \dfrac{1}{4}, \quad \lim_{\alpha_d \to 0} p_d(\alpha_d) = \dfrac{3}{4}; \quad d = 1, \\[2mm] \lim_{\alpha_d \to 1} p_d(\alpha_d) = 0, \quad \lim_{\alpha_d \to 0} p_d(\alpha_d) = \dfrac{1}{2}; \quad d = 2, \\[2mm] \lim_{\alpha_d \to \frac{1}{2}} p_d(\alpha_d) = 0, \quad \lim_{\alpha_d \to 0} p_d(\alpha_d) = \dfrac{1}{4}; \quad d = 3. \end{cases}$$

On any compact time interval $\mathbb{T} = [0, T]$, the inequality (2.20) may thus be rewritten as

$$(2.22) \quad \begin{cases} \int_0^t \sum_{x \in \mathbb{X}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTRW}_\delta^d}\right]^2 ds \leq C_d \left(\sup_{\alpha_d \in I_d} T^{p_d(\alpha_d)}\right) \delta^d |z|^{2\alpha_d} \\ \hspace{4cm} = \tilde{C}_d \delta^d |z|^{2\alpha_d}; \\ \int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTBM}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTBM}_\delta^d}\right]^2 dxds \leq \tilde{C}_d |z|^{2\alpha_d}; \end{cases}$$

$\alpha_d \in I_d$, with $\tilde{C}_d < \infty$ in (2.22). We emphasize that, while we may take $\alpha_1 = 1$ in (2.20) or (2.22); we can neither take the limiting values $\alpha_2 = 1$ nor $\alpha_3 = \frac{1}{2}$ since $\lim_{\alpha_2 \to 1} C_2 = +\infty = \lim_{\alpha_3 \to 1/2} C_3$.

Proof. Starting with the L^2 estimate involving the spatial difference of the BTBM

density in (2.20), we have

$$\int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTBM}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTBM}_\delta^d} \right]^2 dxds$$

$$= 4 \int_0^t \left[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \prod_{i=1}^2 \left(K_{r_i;x}^{\mathrm{BM}_\delta^d} - K_{r_i;x+z}^{\mathrm{BM}_\delta^d} \right) K_{s;0,r_i}^{\mathrm{BM}} dx dr_1 dr_2 \right] ds$$

$$= 4 \int_0^t \int_0^\infty \int_0^\infty \left(2K_{r_1+r_2;0}^{\mathrm{BM}_\delta^d} - 2K_{r_1+r_2;z}^{\mathrm{BM}_\delta^d} \right) K_{s;0,r_1}^{\mathrm{BM}} K_{s;0,r_2}^{\mathrm{BM}} dr_1 dr_2 ds$$

$$= 8 \int_0^t \int_0^\infty \int_0^\infty \frac{1 - e^{-\frac{|z|^2}{2(r_1+r_2)}}}{[2\pi(r_1+r_2)]^{d/2}} K_{s;0,r_1}^{\mathrm{BM}} K_{s;0,r_2}^{\mathrm{BM}} dr_1 dr_2 ds$$

(2.23)
$$= 8 \int_0^t \int_0^{\pi/2} \int_0^\infty \frac{\left(1 - e^{-\frac{|z|^2}{2\rho(\sin(\theta)+\cos(\theta))}} \right) e^{-\rho^2/2s}}{(2\pi)^{\frac{d+2}{2}} \left[\rho(\sin(\theta)+\cos(\theta)) \right]^{d/2} s} \rho d\rho d\theta ds$$

$$\leq C \int_0^t \int_0^\infty \frac{\left(1 - e^{-\frac{|z|^2}{2\rho}} \right) e^{-\rho^2/2s}}{\rho^{d/2} s} \rho d\rho ds$$

$$\leq C \int_0^t \int_0^\infty \frac{|z|^{2\alpha} e^{-\rho^2/2s}}{\rho^{\alpha+d/2} s} \rho d\rho ds$$

$$\leq \begin{cases} C_1 |z|^{2\alpha} t^{p_1(\alpha)}; & d=1, \alpha \in (0,1], \\ C_2 |z|^{2\alpha} t^{p_2(\alpha)}; & d=2, \alpha \in (0,1), \\ C_3 |z|^{2\alpha} t^{p_3(\alpha)}; & d=3, \alpha \in (0,\frac{1}{2}), \end{cases}$$

where we have used the facts that $\min_{0 \leq \theta \leq \pi/2} [\sin(\theta) + \cos(\theta)] = 1$ and that $1 - e^{-u} \leq u^\alpha$ for $u \geq 0$ and $0 < \alpha \leq 1$. This proves the L^2 estimate for the BTBM density in (2.20). Then, an asymptotic argument similar to the one in the proofs of Lemma 2.2 and Lemma 2.3 yields

$$(2.24) \quad \lim_{\delta \searrow 0} \int_0^t \sum_{x \in \mathbb{X}^d} \frac{\left[\mathbb{K}_{s;x}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTRW}_\delta^d} \right]^2}{\delta^d} ds = \int_0^t \int_{\mathbb{R}^d} \left[\mathbb{K}_{s;x}^{\mathrm{BTBM}_\delta^d} - \mathbb{K}_{s;x+z}^{\mathrm{BTBM}_\delta^d} \right]^2 dxds,$$

together with the desired BTRW density L^2 estimate in (2.20) for all $\delta \leq \delta^*$, for some $\delta^* > 0$, with possibly different constants. $\quad\square$

2.2. Estimates for BTRW SIEs and their limits: regularity and tightness.
In this subsection, and assuming only the less-than-Lipschitz conditions (NLip) on a, we obtain spatial and temporal differences moments estimates that yield the regularity of the BTRW SIE $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ for each fixed $n \in \mathbb{N}^*$ (see (2.1)), the tightness of the BTRW SIEs $\left\{ e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n) \right\}_{n \in \mathbb{N}^*}$, as well as the Hölder regularity for the limit.

Fix $n \in \mathbb{N}^*$, and assume $\tilde{U}_n^x(t)$ solves $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ in (1.11). Suppressing the dependence on n, let $M_q(t) = \sup_x \mathbb{E}|\tilde{U}^x(t)|^{2q}$, $q \geq 1$. Writing $\tilde{U}^x(t)$ in terms of its deterministic and random parts $\tilde{U}^x(t) = \tilde{U}_D^x(t) + \tilde{U}_R^x(t)$, we observe that the

deterministic part $\tilde{U}_D^x(t)$ is smooth in time by Lemma 1.1. The next two lemmas give us estimates on the random part.

Lemma 2.5 (Spatial differences). *Assume that* (NLip) *holds and that $M_q(t)$ is bounded on any time interval $\mathbb{T} = [0, T]$. There exists a constant C_d depending only on q, $\max_x |u_0(x)|$, the spatial dimension $1 \le d \le 3$, α_d, and T such that*

$$\mathbb{E} \left| \tilde{U}_R^x(t) - \tilde{U}_R^y(t) \right|^{2q} \le C_d |x - y|^{2q\alpha_d}; \ \alpha_d \in I_d,$$

for all $x, y \in \mathbb{X}^d$, $t \in \mathbb{T}$, and $1 \le d \le 3$; where α_d and I_d are as in Lemma 2.4. I.e., in $d = 1$, we may take $\alpha_1 = 1$; in $d = 2$ we may take any fixed $\alpha_2 \in (0, 1)$; and in $d = 3$, α_3 may be taken to be any fixed value in $(0, \frac{1}{2})$.

Proof.   Using Burkholder inequality, we have for any $(t, x, y) \in \mathbb{T} \times \mathbb{X}^{2d}$

$$(2.25) \quad \mathbb{E} \left| \tilde{U}_R^x(t) - \tilde{U}_R^y(t) \right|^{2q} \le C \mathbb{E} \left| \sum_{z \in \mathbb{X}^d} \int_0^t \left[\mathbb{K}_{t-s;x,z}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{t-s;y,z}^{\mathrm{BTRW}_\delta^d} \right]^2 a^2(\tilde{U}^z(s)) \frac{ds}{\delta^d} \right|^q$$

For any fixed but arbitrary point $(t, x, y) \in \mathbb{T} \times \mathbb{X}^{2d}$ let $\mu_t^{x,y}$ be the measure defined on $[0, t] \times \mathbb{X}^d$ by

$$d\mu_t^{x,y}(s, z) = \left[\mathbb{K}_{t-s;x,z}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{t-s;y,z}^{\mathrm{BTRW}_\delta^d} \right]^2 \frac{ds}{\delta^d},$$

and let $|\mu_t^{x,y}| = \mu_t^{x,y}([0, t] \times \mathbb{X}^d)$. We see from (2.25), Jensen's inequality applied to the probability measure $\mu_t^{x,y} / |\mu_t^{x,y}|$, the growth condition on a, the definition of $M_q(t)$, and elementary inequalities, that we have

$$(2.26) \quad \begin{aligned} \mathbb{E} \left| \tilde{U}_R^x(t) - \tilde{U}_R^y(t) \right|^{2q} &\le C \mathbb{E} \left[\int_{[0,t] \times \mathbb{X}^d} \left| a(\tilde{U}^z(s)) \right|^{2q} \frac{d\mu_t^{x,y}(s, z)}{|\mu_t^{x,y}|} \right] |\mu_t^{x,y}|^q \\ &\le C \left[\int_{[0,t] \times \mathbb{X}^d} (1 + M_q(s)) \frac{d\mu_t^{x,y}(s, z)}{|\mu_t^{x,y}|} \right] |\mu_t^{x,y}|^q \end{aligned}$$

Now, using the boundedness assumption on M_q on $\mathbb{T}$ for $1 \le d \le 3$, we get

$$\mathbb{E} \left| \tilde{U}_R^x(t) - \tilde{U}_R^y(t) \right|^{2q} \le C |\mu_t^{x,y}|^q \le \left[C_d t^{p_d(\alpha_d)} \right]^q |x - y|^{2q\alpha_d} \le \tilde{C}_d |x - y|^{2q\alpha_d}; \ \alpha_d \in I_d,$$

where the last inequality follows from Lemma 2.4 and (2.22) in Remark 2.2, and where the constant $\tilde{C}_d = \left[C_d \sup_{\alpha_d \in I_d} T^{p_d(\alpha_d)} \right]^q < \infty$. $\quad \square$

Lemma 2.6 (Temporal differences). *Assume that* (NLip) *holds and that $M_q(t)$ is bounded on any time interval $\mathbb{T} = [0, T]$. There exists a constant C depending only on q, $\max_x |u_0(x)|$, the spatial dimension $1 \le d \le 3$, and T such that*

$$\mathbb{E} \left| \tilde{U}_R^x(t) - \tilde{U}_R^x(r) \right|^{2q} \le C |t - r|^{\frac{(4-d)q}{4}},$$

for all $x \in \mathbb{X}^d$, for all $t, r \in \mathbb{T}$, and for $1 \le d \le 3$.

Proof.   Assume without loss of generality that $r < t$. Using Burkholder inequality, and using the change of variable $\rho = t - s$, we have for $(r, t, x) \in \mathbb{T}^2 \times \mathbb{X}^d$

$$
\mathbb{E}\left|\tilde{U}_R^x(t) - \tilde{U}_R^x(r)\right|^{2q} \leq C\mathbb{E}\left|\sum_{z \in \mathbb{X}^d} \int_0^r \left[\mathbb{K}_{t-s;x,z}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{r-s;x,z}^{\mathrm{BTRW}_\delta^d}\right]^2 a^2(\tilde{U}^z(s))\frac{ds}{\delta^d}\right|^q
$$

(2.27)

$$
+ C\mathbb{E}\left|\sum_{z \in \mathbb{X}^d} \int_0^{t-r} \left[\mathbb{K}_{\rho;x,z}^{\mathrm{BTRW}_\delta^d}\right]^2 a^2(\tilde{U}^z(\rho))\frac{d\rho}{\delta^d}\right|^q
$$

For a fixed point (r, t, x), let $\mu_{t,r}^x$ be the measure defined on $[0, r] \times \mathbb{X}^d$ by

$$
d\mu_{t,r}^x(s, z) = \left[\mathbb{K}_{t-s;x,z}^{\mathrm{BTRW}_\delta^d} - \mathbb{K}_{r-s;x,z}^{\mathrm{BTRW}_\delta^d}\right]^2 \frac{ds}{\delta^d}
$$

and let $|\mu_{t,r}^x| = \mu_{t,r}^x([0, t] \times \mathbb{X}^d)$. Also, for a fixed $x \in \mathbb{X}^d$, let κ^x be the measure defined on $[0, t - r] \times \mathbb{X}^d$ by

$$
d\kappa^x(\rho) = \left[\mathbb{K}_{\rho;x,z}^{\mathrm{BTRW}_\delta^d}\right]^2 \frac{d\rho}{\delta^d}
$$

and let $|\kappa^x| = \kappa^x([0, t - r] \times \mathbb{X}^d)$. Then, arguing as in Lemma 2.5 above we get that

$$
\mathbb{E}\left|\tilde{U}_R^x(t) - \tilde{U}_R^x(r)\right|^{2q} \leq C\left(\left|\mu_{t,r}^x\right|^q + |\kappa^x|^q\right) \leq C(t - r)^{\frac{(4-d)q}{4}},
$$

for $1 \leq d \leq 3$, where the last inequality follows from Lemma 2.2 and Lemma 2.3, completing the proof. $\qquad\square$

Remark 2.3. In Theorem B.1 of Appendix B we directly (without discretization) prove that, under the Lipschitz conditions (Lip), our BTP SIE in (1.3) has an $L^p(\Omega)$-bounded solution $U(t, x)$ on $\mathbb{T} \times \mathbb{R}^d$ ($M_q(t)$ is bounded on any time interval $\mathbb{T}$) for any $T > 0$ and any $p \geq 2$. We can then repeat the same arguments in the proofs of Lemma 2.5 and Lemma 2.6 above with obvious modifications (replace $\tilde{U}$ with U, sums over $\mathbb{X}^d$ with integrals over $\mathbb{R}^d$ (with $dsdz$ instead of ds/δ^d or $d\rho/\delta^d$), and $\mathbb{K}^{\mathrm{BTRW}_\delta^d}$ with $\mathbb{K}^{\mathrm{BTBM}^d}$; and use the BTBM statements in Lemma 2.2, Lemma 2.3, and Lemma 2.4) to get the same estimates on the spatial and temporal differences of U, with possibly different constants. This in turn straightforwardly leads to the same local Hölder regularity for the direct solution of $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$, U, as the one stated and proved for the BTRW limit solutions in Theorem 1.2.

It is easily seen that if a is bounded then, for all spatial dimensions $1 \leq d \leq 3$, M_q is bounded on any compact time interval $\mathbb{T} = [0, T]$ (see Remark 2.4 below). The following Proposition gives an exponential upper bound on the growth of M_q in time in all $1 \leq d \leq 3$ under the conditions in (NLip).

Proposition 2.1 (Exponential bound for M_q). *Assume that $\tilde{U}^x(t)$ is a solution of the BTRW SIE $e_{BTRW}^{SIE}(a, u_0, n)$ on $\mathbb{T} \times \mathbb{X}^d$, and assume that conditions (NLip) are satisfied. There exists a constant C depending only on q, $\max_x |u_0(x)|$, the dimension d, and T such that*

$$
M_q(t) \leq C\left(1 + \int_0^t M_q(s)ds\right); \quad \forall 0 \leq t \leq T, \ q \geq 1, \ and \ 1 \leq d \leq 3,
$$

and hence

$$M_q(t) \le C \exp\{Ct\}; \quad \forall\, 0 \le t \le T, \ q \ge 1, \ and \ 1 \le d \le 3.$$

In particular, M_q is bounded on $\mathbb{T}$ for all $q \ge 1$ and $1 \le d \le 3$.

The proof of Proposition 2.1 proceeds via the following lemma and its corollary.

Lemma 2.7. *Under the same assumptions as in Proposition 2.1 there exists a constant C depending only on $q \ge 1$, $\max_x |u_0(x)|$, the dimension d, and T such that*

$$M_q(t) \le \begin{cases} C\left(1 + \displaystyle\int_0^t \frac{M_q(s)}{(t-s)^{d/4}}ds\right); & 0 < t \le T, \ q \ge 1, \ and \ 1 \le d \le 3, \\[2mm] C; & t = 0, \ q \ge 1, \ and \ 1 \le d \le 3. \end{cases}$$

Proof.   Fix $q \ge 1$, let $\tilde{U}_D^x(t) \overset{\triangle}{=} \sum\limits_{y \in \mathbb{X}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_\delta^d} u_0(y)$ (the deterministic part of $\tilde{U}$).

Then, for any $(t,x) \in \mathbb{T} \times \mathbb{X}^d$, we apply Burkholder inequality to the random term $\tilde{U}_R^x(t)$ to get

$$\mathbb{E}\left|\tilde{U}^x(t)\right|^{2q} = \mathbb{E}\left|\sum_{y \in \mathbb{X}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d} \frac{a(\tilde{U}^y(s))}{\delta^{d/2}} dW^y(s) + \tilde{U}_D^x(t)\right|^{2q}$$

(2.28)

$$\le C\left(\mathbb{E}\left|\sum_{y \in \mathbb{X}^d} \int_0^t \left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right)^2 \frac{a^2(\tilde{U}^y(s))}{\delta^d} ds\right|^q + \left|\tilde{U}_D^x(t)\right|^{2q}\right).$$

Now, for a fixed point $(t,x) \in \mathbb{T} \times \mathbb{X}^d$ let μ_t^x be the measure on $[0,t] \times \mathbb{X}^d$ defined by $d\mu_t^x(s,y) = \left[\left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right)^2 / \delta^d\right] ds$, and let $|\mu_t^x| = \mu_t^x([0,t] \times \mathbb{X}^d)$. Then, we can rewrite (2.28) as

$$(2.29) \quad \mathbb{E}\left|\tilde{U}^x(t)\right|^{2q} \le C\left(\mathbb{E}\left|\int_{[0,t] \times \mathbb{X}^d} a^2(\tilde{U}^y(s)) \frac{d\mu_t^x(s,y)}{|\mu_t^x|}\right|^q |\mu_t^x|^q + |\tilde{U}_D^x(t)|^{2q}\right).$$

Observing that $\mu_t^x / |\mu_t^x|$ is a probability measure, we apply Jensen's inequality, the growth condition on a in (NLip), and other elementary inequalities to (2.29) to obtain

$$\mathbb{E}\left|\tilde{U}^x(t)\right|^{2q} \le C\left(\mathbb{E}\left[\int_{[0,t] \times \mathbb{X}^d} \left|a(\tilde{U}^y(s))\right|^{2q} \frac{d\mu_t^x(s,y)}{|\mu_t^x|}\right] |\mu_t^x|^q + \left|\tilde{U}_D^x(t)\right|^{2q}\right)$$

$$\le C\left[\int_{[0,t] \times \mathbb{X}^d} \left(1 + \mathbb{E}\left|\tilde{U}^y(s)\right|^{2q}\right) d\mu_t^x(s,y)\right] |\mu_t^x|^{q-1} + C\left|\tilde{U}_D^x(t)\right|^{2q}$$

$$= C\left(\left[\sum_{y \in \mathbb{X}^d} \int_0^t \frac{\left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right)^2}{\delta^d} \left(1 + \mathbb{E}\left|\tilde{U}^y(s)\right|^{2q}\right) ds\right] |\mu_t^x|^{q-1} + \left|\tilde{U}_D^x(t)\right|^{2q}\right)$$

Using Lemma 2.2 we see that $|\mu_t^x|$ is uniformly bounded for $t \le T$ and $1 \le d \le 3$. So, using the boundedness of u_0, and hence of $\tilde{U}_D^x(t)$ by the simple fact that $\sum_{y \in \mathbb{X}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_\delta^d} = 1$, Lemma 2.2 and the definition of $M_q(s)$, we get

$$\mathbb{E}\left|\tilde{U}^x(t)\right|^{2q} \le C\left(1 + \sum_{y \in \mathbb{X}^d} \int_0^t \frac{\left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right)^2}{\delta^d} M_q(s)ds\right)$$

$$\overset{R_1}{\le} C\left(1 + \int_0^t \frac{M_q(s)}{(t-s)^{d/4}}ds\right).$$

Here, R_1 holds for $1 \le d \le 3$. This implies that

$$M_q(t) \le C\left(1 + \int_0^t \frac{M_q(s)}{(t-s)^{d/4}}ds\right).$$

Of course, $M_q(0) = \sup_x |u_0(x)|^{2q} \le C$, by the boundedness and nonrandomness assumptions on $u_0(x)$ in (NLip). The proof is complete. $\square$

Remark 2.4. It is clear that for a bounded a, M_q is locally bounded in time. This follows immediately from Lemma 2.2 along with (2.29) above.

Corollary 2.1. *Under the same assumptions as those in Proposition 2.1 there exists a constant C depending only on q, $\max_x |u_0(x)|$, the dimension d, and T such that*

$$M_q(t) \le C\left(1 + \int_0^t M_q(s)ds\right), 1 \le d \le 3;$$

and hence

$$M_q(t) \le C \exp\{Ct\}; \quad \forall 0 \le t \le T, \; q \ge 1, \; and \; 1 \le d \le 3.$$

Proof.   Iterating the bound in Lemma 2.7 once, and changing the order of integration, we obtain

(2.30)

$$M_q(t) \le C\left\{1 + C\left[\int_0^t \frac{ds}{(t-s)^{d/4}} + \int_0^t M_q(r)\left(\int_r^t \frac{ds}{(t-s)^{d/4}(s-r)^{d/4}}\right)dr\right]\right\}$$

$$\le C\left(1 + \int_0^t M_q(s)ds\right)$$

for $1 \le d \le 3$. The proof of the last statement is a straightforward application of Gronwall's lemma to (2.30). This finishes the proof of Corollary 2.1 and thus of Proposition 2.1. $\square$

The regularity, tightness, and weak limit conclusions for the BTRW SIEs now easily follow as

Lemma 2.8 (Regularity and tightness). *Assume that the conditions* (NLip) *hold, and that* $\left\{\tilde{U}_n^x(t)\right\}_{n \in \mathbb{N}^*}$ *is a sequence of spatially-linearly-interpolated solutions to the BTRW SIEs* $\left\{e_{BTRW}^{SIE}(a, u_0, n)\right\}_{n \in \mathbb{N}^*}$ *in* (1.11). *Then*

(a) *For every* n, $\tilde{U}_n^x(t)$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}^d$. *Moreover, with probability one, the continuous map* $(t, x) \mapsto \tilde{U}_n^x(t)$ *is locally* γ_t-*Hölder continuous in time with* $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ *for* $1 \leq d \leq 3$.

(b) *There is a BTRW SIE weak limit solution to* $e_{BTP}^{SIE}(a, u_0)$, *call it* U, *such that* $U(t, x)$ *is* $L^p(\Omega, \mathbb{P})$-*bounded on* $\mathbb{T} \times \mathbb{R}^d$ *for every* $p \geq 2$ *and* $U \in \mathrm{H}^{\frac{4-d}{8}, \alpha_d}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ *for every* $1 \leq d \leq 3$ *and* $\alpha_d \in I_d$, *where* α_d *and* I_d *are as in Theorem 1.2 (also Lemma 2.4).*

Remark 2.5. Of course in part (a) above, even without linear interpolation in space, $\tilde{U}^x(t)$ is locally Hölder continuous in time with Hölder exponent $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.

Proof. For each n, let $\tilde{U}_n^x(t) = \tilde{U}_{n,D}^x(t) + \tilde{U}_{n,R}^x(t)$ be the decomposition of $\tilde{U}_n^x(t)$ in (1.11) into its deterministic and random parts, respectively.

(a) By Lemma 1.1, $\tilde{U}_{n,D}^x(t)$ is clearly smooth in time; so it is enough to consider the random term $\tilde{U}_{n,R}^x(t)$. We let $q_m = m + 2$ for $m \in \{0, 1, \ldots\}$, we then have from Lemma 2.6 that

$$(2.31) \qquad \mathbb{E}\left|\tilde{U}_R^x(t) - \tilde{U}_R^x(r)\right|^{4+2m} \leq C \left|t - r\right|^{\frac{(4-d)(m+2)}{4}}.$$

for $1 \leq d \leq 3$. Thus as in Theorem 2.8 p. 53 [25] we get that $\gamma_t \in \left(0, \frac{(4 + (4-d)m - 2d)/4}{2m+4}\right)$ for every m. Taking the limit as $m \to \infty$, we get $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.

(b) By Lemma 2.1 it follows that $\tilde{U}_{n,D}^x(t)$ converges pointwise to the deterministic part of $e_{BTP}^{SIE}(a, u_0)$ in (1.3); i.e.,

$$(2.32) \qquad \lim_{n \to \infty} \tilde{U}_{n,D}^x(t) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{BTBM^d} u_0(y) dy.$$

We also conclude from Lemma 2.5 and Lemma 2.6 that the sequence $\left\{\tilde{U}_{n,R}^x(t)\right\}_{n \in \mathbb{N}^*}$ is tight on $\mathrm{C}(\mathbb{T} \times \mathbb{R}^d)$ for $1 \leq d \leq 3$. Thus there exists a weakly convergent subsequence $\left\{\tilde{U}_{n_k}\right\}_{k \in \mathbb{N}}$ and hence a BTRW SIE weak limit solution U to $e_{BTP}^{SIE}(a, u_0)$. Then, following Skorokhod, we can construct processes $Y_k \overset{d}{=} \tilde{U}_{n_k}$ on some probability space $(\Omega^S, \mathscr{F}^S, \{\mathscr{F}_t^S\}, \mathbb{P}^S)$ such that with probability 1, as $k \to \infty$, $Y_k(t, x)$ converges to a random field $Y(t, x)$ uniformly on compact subsets of $\mathbb{T} \times \mathbb{R}^d$ for $1 \leq d \leq 3$. Now, for the BTRW SIEs limit regularity assertions, clearly the deterministic term on the right hand side of (2.32) is $\mathrm{C}^{1,4}$ and bounded as in [9, 8], so we use Proposition 2.1, Lemma 2.5, and Lemma 2.6 to obtain the regularity results for the random part. We provide the steps here for completeness.

First, $Y_k \overset{d}{=} \tilde{U}_{n_k}$ and so Proposition 2.1 gives us, for each $p \geq 2$:

(2.33) $\mathbb{E}\,|Y_k(t,x)|^p = \mathbb{E}\left|\tilde{U}_{n_k}^x(t)\right|^p \leq C < \infty; \forall (t,x,k) \in \mathbb{T} \times \mathbb{R}^d \times \mathbb{N}, 1 \leq d \leq 3,$

for some constant C that is independent of k, t, x but that depends on the dimension d. It follows that, for each $(t,x) \in \mathbb{T} \times \mathbb{R}^d$ the sequence $\left\{|Y_k(t,x)|^p\right\}_k$ is uniformly integrable for each $p \geq 2$ and each $1 \leq d \leq 3$. Thus,

(2.34) $\mathbb{E}\,|U(t,x)|^p = \mathbb{E}\,|Y(t,x)|^p = \lim_{k\to\infty} \mathbb{E}\,|Y_k(t,x)|^p \leq C < \infty; \; \forall (t,x) \in \mathbb{T} \times \mathbb{R}^d,$

for all $1 \leq d \leq 3$ and $p \geq 2$. Equation (2.34) establishes the L^p boundedness assertion. In addition, for $q \geq 1$ and $1 \leq d \leq 3$ we have by Proposition 2.1

(2.35)
$$\mathbb{E}\,|Y_k(t,x) - Y_k(t,y)|^{2q} + \mathbb{E}\,|Y_k(t,x) - Y_k(r,x)|^{2q}$$
$$\leq C\left[\mathbb{E}\,|Y_k(t,x)|^{2q} + \mathbb{E}\,|Y_k(t,y)|^{2q} + \mathbb{E}\,|Y_k(r,x)|^{2q}\right]$$
$$\leq C; \; \forall (k,r,t,x,y) \in \mathbb{N} \times \mathbb{T}^2 \times \mathbb{R}^2.$$

So, for each $(r,t,x,y) \in \mathbb{T}^2 \times \mathbb{R}^2$, the sequences $\left\{|Y_k(t,x) - Y_k(t,y)|^{2q}\right\}_k$ and $\left\{|Y_k(t,x) - Y_k(r,x)|^{2q}\right\}_k$ are uniformly integrable, for each $q \geq 1$. Therefore, using Lemma 2.5 and Lemma 2.6, we obtain

(2.36)
$$\begin{cases} \mathbb{E}\,|U(t,x) - U(t,y)|^{2q} = \mathbb{E}\,|Y(t,x) - Y(t,y)|^{2q} \\ = \lim_{k\to\infty} \mathbb{E}\,|Y_k(t,x) - Y_k(t,y)|^{2q} \leq C_d |x-y|^{2q\alpha_d}; \; \alpha_d \in I_d, \\ \mathbb{E}\,|U(t,x) - U(r,x)|^{2q} = \mathbb{E}\,|Y(t,x) - Y(r,x)|^{2q} \\ = \lim_{k\to\infty} \mathbb{E}\,|Y_k(t,x) - Y_k(r,x)|^{2q} \leq C\,|t-r|^{\frac{(4-d)q}{4}}. \end{cases}$$

Now, for $d = 1, 2, 3$, we let $q_n = n + d$ for $n \in \{0,1,\ldots\}$ and let $n = m + d$ for $m = \{0,1,\ldots\}$, we then have from (2.36) that

(2.37)
$$\begin{cases} \mathbb{E}\,|U(t,x) - U(t,y)|^{2n+2d} \leq C_d\,|x-y|^{(2n+2d)\alpha_d}, \\ \mathbb{E}\,|U(t,x) - U(r,x)|^{2m+4d} \leq C\,|t-r|^{\frac{(4-d)(m+2d)}{4}}. \end{cases}$$

for $1 \leq d \leq 3$. Thus as in Theorem 2.8 p. 53 and Problem 2.9 p. 55 in [25] we get that the spatial Hölder exponent is $\gamma_s \in \left(0, \frac{2(n+d)\alpha_d - d}{2n+2d}\right)$ and the temporal exponent is $\gamma_t \in \left(0, \frac{m(1-d/4)+d(2-d/2)-1}{2m+4d}\right) \forall m, n$. Taking the limits as $m, n \to \infty$, we get $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ and $\gamma_s \in (0, \alpha_d)$, for $1 \leq d \leq 3$.
The proof is complete. $\square$

2.3. From K-martingale problems to truncated BTRW SIEs. Here we give the

Proof of Theorem 1.1.   Assume that $\left(\left\{X_{n,l}^\tau\right\}_{\tau \geq 0}, \{W_n^y(t)\}_{y \in \mathbb{X}_{n,l}^d}\right)$ is a solution to $\left\{e_{\mathrm{BTRW}}^{\mathrm{aux\text{-}SIE}}(a, u_0, n, l, \tau)\right\}_{\tau \geq 0}$ in (Aux) on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$; then clearly $X_{n,l}^{\tau,x}(0) = u_0(x)$ for $x \in \mathbb{X}_n^d$ and $X_{n,l}^{\tau,x}(t) = \tilde{U}_{n,D}^x(t)$ for $x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d$,

for every $\tau \geq 0$ and $t \in [0, \tau]$. Fix any arbitrary $\tau_1, \tau_2 > 0$ and $x_1, x_2 \in \mathbb{X}_{n,l}^d$. By Itô's formula we have

$$
f(Z_{n,l}^{x_1,2,\tau_1,2}(t)) - f(u_0^{x_1,x_2}) - \int_0^t \left(\mathscr{A}_\Upsilon^{\tau_1,2} f \right)(s, x_1, x_2, Z_{n,l}^{\tau_1,2}) ds
$$

(2.38)
$$
= \sum_{i=1}^2 \sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \partial_i f(Z_{n,l}^{x_1,2,\tau_1,2}(s)) \kappa_{\delta_n,s,\tau_i}^{x_i,y} \left(X_{n,l}^{\tau_i,y}(s) \right) dW_n^y(s) \in \mathscr{M}_2^{c,loc}
$$

for $t \in [0, \tau_1 \wedge \tau_2]$ and (KM) is satisfied.

Conversely, if a family of adapted

$$
\left\{ X_{n,l}^\tau \right\}_{\tau \geq 0} = \left\{ \left\{ X_{n,l}^{\tau,x}(t); 0 \leq t \leq \tau, x \in \mathbb{X}_n^d \right\} \right\}_{\tau \geq 0}
$$

defined on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ satisfies (KM); then fixing any two $\tau_1, \tau_2 > 0$, letting $\mathbb{B}_{0,R} = \left\{ u = (u_1, u_2) \in \mathbb{R}^2; |u| \leq R \right\}$ and choosing $f_1, f_2, f \in C_b^2(\mathbb{R}; \mathbb{R})$ such that $f_i(u) = u_i$ for $i = 1, 2$ and $f(u) = u_1 u_2$ whenever $u \in \mathbb{B}_{0,R}$ we see that

(2.39)
$$
\begin{cases}
M^{\tau_i,x_i}(t) \quad := X_{n,l}^{\tau_i,x_i}(t) - \tilde{U}_{n,D}^{x_i}(t) \in \mathscr{M}_2^{c,loc}; i = 1, 2, \\[2mm]
N^{\tau_1,2,x_1,2}(t) := \prod_{i=1}^2 X_{n,l}^{\tau_i,x_i}(t) - \prod_{i=1}^2 u_0(x_i) \\[2mm]
\qquad\qquad - \sum_{\substack{1 \leq i,j \leq 2 \\ i \neq j}} \int_0^t X_{n,l}^{\tau_i,x_i}(s) d\tilde{U}_{n,D}^{x_j}(s) \\[2mm]
\qquad\qquad - \int_0^t \sum_{y \in \mathbb{X}_{n,l}^d} \Upsilon_{\delta_n,t,\tau_1,2}^{x_1,2,y} \left(Z_{n,l}^{y,\tau_1,2}(s) \right) ds \in \mathscr{M}_2^{c,loc}
\end{cases}
$$

for all $t \in [0, \tau_1 \wedge \tau_2]$ and $x_1, x_2 \in \mathbb{X}_{n,l}^d$, where we have used the notation $Z_{n,l}^{\tau_1,2}$ for the two-dimensional process defined in (1.19). We then have that

(2.40)
$$
\prod_{i=1}^2 M^{\tau_i,x_i}(t) - \sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \Upsilon_{\delta_n,t,\tau_1,2}^{x_1,2,y} \left(Z_{n,l}^{y,\tau_1,2}(s) \right) ds
$$
$$
= T_1^{x_1,2,\tau_1,2}(t) + T_2^{x_1,2,\tau_1,2}(t)
$$

where

(2.41)
$$
T_1^{x_1,2,\tau_1,2}(t) := N^{\tau_1,2,x_1,2}(t) - \sum_{\substack{1 \leq i,j \leq 2 \\ i \neq j}} u_0(x_i) M^{\tau_j,x_j}(t) \in \mathscr{M}_2^{c,loc}
$$

and

(2.42)
$$
T_2^{x_1,2,\tau_1,2}(t) := \sum_{\substack{1 \leq i,j \leq 2 \\ i \neq j}} \int_0^t \left[X_{n,l}^{\tau_i,x_i}(s) - X_{n,l}^{\tau_i,x_i}(t) \right] d\tilde{U}_{n,D}^{x_j}(s) + \prod_{i=1}^2 \left[\tilde{U}_{n,D}^{x_i}(t) - u_0(x_i) \right]
$$
$$
= \sum_{\substack{1 \leq i,j \leq 2 \\ i \neq j}} \int_0^t \left[u_0(x_i) - \tilde{U}_{n,D}^{x_i}(u) \right] dM^{\tau_j,x_j}(u) \in \mathscr{M}_2^{c,loc}.
$$

Thus,

$$(2.43) \qquad \langle M^{\tau_1,x_1}(\cdot), M^{\tau_2,x_2}(\cdot)\rangle_t = \sum_{y\in\mathbb{X}_{n,l}^d} \int_0^t \Upsilon_{\delta_n,t,\tau_{1,2}}^{x_{1,2},y} \left(Z_{n,l}^{y,\tau_{1,2}}(s) \right) ds.$$

Equations (2.39) and (2.43) imply that there exists a set of $r = \# \left\{ y; y \in \mathbb{X}_{n,l}^d \right\}$ independent Brownian motions $\{W_n^y(t); t \in \mathbb{R}_+\}_{y\in\mathbb{X}_{n,l}^d}$ on an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \{\tilde{\mathscr{F}}_t\}, \tilde{\mathbb{P}})$ such that

$$(2.44)\ \ M^{\tau,x}(t) = \sum_{y\in\mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n,s,\tau}^{x,y} \left(X_{n,l}^{\tau,y}(s) \right) dW_n^y(s),\ \forall (x,\tau,t) \in \mathbb{X}_{n,l}^d \times \mathbb{R}_+ \times [0,\tau].$$

In fact, fixing any $\tau > 0$ and labeling the $\left\{ x; x \in \mathbb{X}_{n,l}^d \right\}$ as $\{x_1,\ldots,x_r\}$, the restriction of the desired family of BMs $\{W_n^y(t); t \in \mathbb{R}_+\}_{y\in\mathbb{X}_{n,l}^d}$ to the time interval $[0,\tau]$ is obtained from the matrix equation (written in differential form)

$$(2.45)$$

$$
\begin{bmatrix} dW_n^{x_1}(t) \\ \vdots \\ dW_n^{x_r}(t) \end{bmatrix} = \begin{bmatrix} \kappa_{\delta_n,t,\tau}^{x_1,x_1}\left(X_{n,l}^{\tau,x_1}(t)\right) & \cdots & \kappa_{\delta_n,t,\tau}^{x_1,x_r}\left(X_{n,l}^{\tau,x_r}(t)\right) \\ \vdots & \vdots & \vdots \\ \kappa_{\delta_n,t,\tau}^{x_r,x_1}\left(X_{n,l}^{\tau,x_1}(t)\right) & \cdots & \kappa_{\delta_n,t,\tau}^{x_r,x_r}\left(X_{n,l}^{\tau,x_r}(t)\right) \end{bmatrix}^{-1} \begin{bmatrix} dM^{\tau,x_1}(t) \\ \vdots \\ dM^{\tau,x_r}(t) \end{bmatrix}
$$

$$
:= \begin{bmatrix} d\tilde{W}_n^{\tau,x_1}(t) \\ \vdots \\ d\tilde{W}_n^{\tau,x_r}(t) \end{bmatrix}
$$

whenever the middle inverse kernel-diffusion coefficient $r \times r$ matrix, denoted by A^{-1}, exists (the determinant $\det(A) \neq 0$) almost surely. If this fails we can proceed similar to the standard finite dimensional SDE case cf. Ikeda and Watanabe [24] or Doob [22]. It is now straightforward to verify that, for any $\tau_1, \tau_2 > 0$ and any $t \in [0, \tau_1 \wedge \tau_2]$, the two families of BMs $\left\{ \left\{ \tilde{W}_n^{\tau_k,y} \right\}_{y\in\mathbb{X}_{n,l}^d}; k = 1, 2 \right\}$ satisfy

$$(2.46) \qquad \langle \tilde{W}_n^{\tau_1,x_i}(\cdot), \tilde{W}_n^{\tau_2,x_j}(\cdot)\rangle_t = \begin{cases} t, & 1 \leq i = j \leq r \\ 0, & 1 \leq i \neq j \leq r \end{cases}$$

almost surely, whether or not $\tau_1 = \tau_2$. I.e., we get one family of independent BMs $\{W_n^y(t)\}_{y\in\mathbb{X}_{n,l}^d}$, such that $\left(\left\{ X_{n,l}^\tau \right\}_{\tau\geq 0}, \{W_n^y\}_{y\in\mathbb{X}_{n,l}^d} \right)$ is a solution pair to $\left\{ e_{\text{BTRW}}^{\text{aux-SIE}}(a, u_0, n, l, \tau) \right\}_{\tau\geq 0}$ in (Aux) on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \{\tilde{\mathscr{F}}_t\}, \tilde{\mathbb{P}})$. Hence, on the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \{\tilde{\mathscr{F}}_t\}, \tilde{\mathbb{P}})$, the pair $\left(\left\{ X_{n,l}^{t,x}(t) \right\}_{t\geq 0, x\in\mathbb{X}_n^d}, \{W_n^y(t)\}_{y\in\mathbb{X}_{n,l}^d} \right)$ solves the l-truncated BTRW SIE $e_{\text{BTRW}}^{\text{t-SIE}}(a, u_0, n, l)$ in (1.13) on $\mathbb{R}_+ \times \mathbb{X}_n^d$. $\qquad\square$

2.4. **BTRW SIEs limits solutions to BTP SIEs: Lipschitz vs. non Lipschitz.** Here, we prove the main result in Theorem 1.2. The type of our BTRW SIE limit solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ in (1.3) depends on the conditions: under the Lipschitz conditions (Lip) we get a direct solution to $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ for every n and a direct BTRW SIE limit solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$; whereas under the non-Lipschitz conditions in (NLip) we obtain a limit BTRW SIE solution, thanks to our K-martingale approach, and a BTRW SIEs double limit solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$.

2.4.1. *The Lipschitz case.* In Section 2.2 we assumed the existence of a BTRW SIE solution and we obtained regularity and tightness for $\left\{ e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n) \right\}_{n \in \mathbb{N}^*}$. This, in turn, implied the existence and regularity for a BTRW SIE limit solution to our $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. To complete the existence for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ it suffices then to prove the existence of a solution to $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ for each fixed $n \in \mathbb{N}^*$, that is uniformly $L^p(\Omega, \mathbb{P})$ bounded on $[0, T] \times \mathbb{X}^d$ for every $T > 0$ and every $p \geq 2$. I.e., Theorem 1.2 (a) follows as a corollary to the results of Section 2.2 combined with the following proposition.

Proposition 2.2. *Under the Lipschitz conditions* (Lip) *there exists a unique direct solution to $e_{BTRW}^{SIE}(a, u_0, n)$, $\tilde{U}$, on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ that is $L^p(\Omega, \mathbb{P})$-bounded on $[0, T] \times \mathbb{X}^d$ for every $T > 0$, $p \geq 2$, and $1 \leq d \leq 3$.*

The proof follows very closely the non-discretization Picard-type direct proof of the continuous case in Appendix B, and we just indicate the minor differences below for the reader's convenience. Also, since the proof works for every lattice $\mathbb{X}_n^d$, $n \in \mathbb{N}^*$, with a uniform $L^p(\Omega)$-bound, we suppress the lattice and the solution dependence on the subscript n; and we use n as the iterate index in the iterative proof below instead.

Proof. For the existence proof, fix a lattice $\mathbb{X}^d$ on which we construct a solution iteratively. So, given a collection of independent BMs $\{W^y(t)\}_{y \in \mathbb{X}^d}$, define for each $(t, x) \in \mathbb{R}_+ \times \mathbb{X}^d$

$$(2.47) \quad \begin{cases} U^{(0)}(t, x) = \displaystyle\sum_{y \in \mathbb{X}^d} \mathbb{K}_{t; x, y}^{\mathrm{BTRW}_\delta^d} u_0(y) \\ U^{(n+1)}(t, x) = U^{(0)}(t, x) + \displaystyle\sum_{y \in \mathbb{X}^d} \int_0^t \mathbb{K}_{t-s; x, y}^{\mathrm{BTRW}_\delta^d} a(U^{(n)}(s, y)) \frac{dW^y(s)}{\delta^{d/2}} \end{cases}$$

Just as in the proof of Theorem B.1 (the BTP SIE case) we show that, for any $p \geq 2$ and all $1 \leq d \leq 3$, the sequence $\{U^{(n)}(t, x)\}_{n \geq 1}$ converges in $L^p(\Omega)$ to a solution. Let

$$D_{n,p}(t, x) := \mathbb{E} \left| U^{(n+1)}(t, x) - U^{(n)}(t, x) \right|^p$$

and

$$M_{n,p}(t) := \sup_x D_{n,p}(t, x).$$

Following the same steps as in the continuous-space BTP SIE case we get

$$(2.48) \quad D_{n,p}^*(t) \leq C \left(t^{\frac{4-d}{4}} \right)^{\frac{p-2}{2}} \int_0^t D_{n-1,p}^*(s) \, [t-s]^{-\frac{d}{4}} \, ds$$

where we have used Lemma 2.2. Again, exactly as in the proof of Theorem B.1, we conclude that there exists an $L^p(\Omega)$-limit $\tilde{U}^x(t)$ that satisfies the BTRW SIE $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ in (1.11) with respect to the given BMs $\{W^y(t)\}_{y \in \mathbb{X}^d}$, and such that $\tilde{U}^x(t)$ is $L^p(\Omega)$ bounded on $\mathbb{T} \times \mathbb{X}^d$, for any $p \geq 2$ and for any $T > 0$. Therefore, by Lemma 2.8 above, that solution (and its spatially-linearly-interpolated version) is almost surely Hölder continuous in time with exponent $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.

Again, the uniqueness proof follows exactly the same steps as its counterpart in Theorem B.1 with the space being $\mathbb{X}^d$ instead of $\mathbb{R}^d$ and with $\mathbb{K}_{t;x,y}^{\mathrm{BTBM}^d}$ replaced with $\mathbb{K}_{t;x,y}^{\mathrm{BTRW}_\delta^d}$. So, let $1 \leq d \leq 3$ and let $\tilde{U}_1$ and $\tilde{U}_2$ be two solutions to $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$ for a fixed n that are $L^2(\Omega)$-bounded on $\mathbb{T} \times \mathbb{X}^d$, for every $T > 0$. Fix an arbitrary $(t, x) \in \mathbb{R}_+ \times \mathbb{X}^d$. Let $D(t, x) = \tilde{U}_2^x(t) - \tilde{U}_1^x(t)$, $L_2(t, x) = \mathbb{E}D^2(t, x)$, and $L_2^*(t) = \sup_{x \in \mathbb{X}^d} L_2(t, x)$ (which is bounded on $\mathbb{T}$ for every $T > 0$ by hypothesis). Then, using (1.11), the Lipschitz condition in (Lip), and taking the supremum over the space variable and using Lemma 2.2 we have

$$L_2(t, x) = \sum_{y \in \mathbb{X}^d} \int_0^t \mathbb{E}\left[a(\tilde{U}_2^y(s)) - a(\tilde{U}_1^y(s))\right]^2 \left[\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right]^2 ds$$

$$(2.49) \qquad \leq C \int_0^t L_2^*(s) \sum_{y \in \mathbb{X}^d} \left[\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_\delta^d}\right]^2 ds$$

$$\leq C \int_0^t \frac{L_2^*(s)}{(t-s)^{\frac{d}{4}}} ds$$

Iterating and interchanging the order of integration we get

$$(2.50) \qquad L_2^*(t) \leq C \left(\int_0^t L_2^*(s) ds\right)$$

for every $t \geq 0$. An easy application of Gronwall's lemma gives that $L_2^* \equiv 0$. So for every $(t, x) \in \mathbb{R}_+ \times \mathbb{X}^d$ and $1 \leq d \leq 3$ we have $\tilde{U}_1^x(t) = \tilde{U}_2^x(t)$ with probability one. So that, using Lemma 2.8 (a), the spatially-linearly-interpolated versions of $\tilde{U}_1^x(t)$ and $\tilde{U}_2^x(t)$ are indistinguishable—and hence pathwise uniqueness holds for such interpolated solutions to $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$—on $\mathbb{R}_+ \times \mathbb{R}^d$. $\quad\square$

Corollary 2.2. *Theorem 1.2 (a) holds.*

Proof. The conclusion follows from Proposition 2.2, Lemma 2.5, Lemma 2.6, and Lemma 2.8 (b). $\quad\square$

2.4.2. *The non-Lipschitz case.* Here, we only assume the less-than-Lipschitz conditions (NLip), and we show the existence of a double limit solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. First, the following proposition summarizes the results in this case for the BTRW SIEs scale.

Proposition 2.3 (Existence for BTRW SIEs without Lipschitz)**.** *Assume the conditions* (NLip) *hold. Then,*

(a) *For every $(n,l) \in \mathbb{N}^* \times \mathbb{N}$ and for every $p \geq 2$, there exists an L^p-bounded solution $\tilde{U}^x_{n,l}(t)$ to the truncated BTRW SIE (1.13) on $\mathbb{T} \times \mathbb{X}^d_n$. Moreover, if we linearly interpolate $\tilde{U}^x_{n,l}(t)$ in space; then, with probability one, the continuous map $(t,x) \mapsto \tilde{U}^x_{n,l}(t)$ is locally γ_t-Hölder continuous in time with $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.*

(b) *For any fixed $n \in \mathbb{N}^*$, the sequence $\left\{\tilde{U}^x_{n,l}(t)\right\}_{l\in\mathbb{N}}$ of linearly-interpolated solutions in (a) has a subsequential weak limit $\tilde{U}_n$ in $\mathrm{C}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$. We thus have a limit solution $\tilde{U}_n$ to $e^{SIE}_{BTRW}(a, u_0, n)$, and $\tilde{U}_n$ is locally γ_t-Hölder continuous in time with $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.*

Proof.

(a) First, recall that the deterministic term $\tilde{U}^x_D(t)$ in (1.13) is completely determined by u_0. Moreover, under the conditions in (NLip) on u_0, $\tilde{U}^x_D(t)$ is clearly bounded and it is smooth in time as in Remark 1.2. Fix an arbitrary $T > 0$, and let $\mathbb{T} = [0, T]$. We now prove the existence of a family of adapted processes $\left\{\tilde{X}^\tau_{n,l}\right\}_{\tau\in\mathbb{T}}$ satisfying our K-martingale problem (KM), which by Theorem 1.1 implies the existence of a solution to the l-truncated BTRW SIE (1.13) on $\mathbb{T} \times \mathbb{X}^d_n$. On a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ we prepare a family of r-independent BMs $\{W^y_n(t)\}_{y\in\mathbb{X}^d_{n,l}}$. For each $\tau \in \mathbb{T}$ and each $i = 1, 2, \dots$ define a continuous process $X^\tau_{n,l,i}$ on $[0, \tau] \times \mathbb{X}^d_n$ inductively for $k/2^i \leq t \leq ((k+1)/2^i) \wedge \tau$ $(k = 0, 1, 2, \dots)$ as follows: $X^{x,\tau}_{n,l,i}(0) = u_0(x)$ $(x \in \mathbb{X}^d_n)$ and if $X^{x,\tau}_{n,l,i}(t)$ is defined for $t \leq k/2^i$, then we define $X^{x,\tau}_{n,l,i}(t)$ for $k/2^i \leq t \leq ((k+1)/2^i) \wedge \tau$, by

(2.51)
$$
X^{x,\tau}_{n,l,i}(t) = \begin{cases} X^{x,\tau}_{n,l,i}\left(\frac{k}{2^i}\right) + \displaystyle\sum_{y\in\mathbb{X}^d_{n,l}} \kappa^{x,y}_{\delta_n, \frac{k}{2^i}, \tau}\left(X^{y,\tau}_{n,l,i}(\tfrac{k}{2^i})\right)\left(\Delta_{t,\frac{k}{2^i}} W^y_n\right) \\ \quad + \left[\tilde{U}^x_{n,D}(t) - \tilde{U}^x_{n,D}\left(\tfrac{k}{2^i}\right)\right]; & x \in \mathbb{X}^d_{n,l}, \\ \tilde{U}^x_{n,D}(t); & x \in \mathbb{X}^d_n \setminus \mathbb{X}^d_{n,l}, \end{cases}
$$

where $\Delta_{t,\frac{k}{2^i}} W^y_n = W^y_n(t) - W^y_n(\frac{k}{2^i})$. Clearly, $X^\tau_{n,l,i}$ is the solution to the equation

(2.52)
$$
X^{x,\tau}(t) = \begin{cases} \displaystyle\sum_{y\in\mathbb{X}^d_{n,l}} \int_0^t \kappa^{x,y}_{\delta_n, \phi_i(s), \tau}\left(X^{y,\tau}(\phi_i(s))\right) dW^y_n(s) + \tilde{U}^x_{n,D}(t); & x \in \mathbb{X}^d_{n,l}, \\ \tilde{U}^x_{n,D}(t); & x \in \mathbb{X}^d_n \setminus \mathbb{X}^d_{n,l} \end{cases}
$$

with $X^{x,\tau}(0) = u_0(x)$, where $\phi_i(t) = k/2^i$ for $k/2^i \leq t < (k+1)/2^i \wedge \tau$ $(k = 0, 1, 2, \dots)$.

Now, for $q \geq 1$, let $M_{q,l,i}^{\tau}(t) = \sup_{x \in \mathbb{X}_n^d} \mathbb{E} \left| X_{n,l,i}^{x,\tau}(t) \right|^{2q}$. By the bounded-ness of $\tilde{U}_{n,D}^x(t)$ over the whole infinite lattice $\mathbb{X}_n^d$, we have

$$(2.53) \qquad M_{q,l,i}^{\tau}(t) \leq C + \sup_{x \in \mathbb{X}_{n,l}^d} \mathbb{E} \left| X_{n,l,i}^{x,\tau}(t) \right|^{2q}$$

Then, replacing $\mathbb{X}_n^d$ by $\mathbb{X}_{n,l}^d$ and following the same steps as in the proof of Proposition 2.1, we get that

$$(2.54) \qquad \sup_{\tau \in \mathbb{T}} \sup_{t \in [0,\tau]} M_{q,l,i}^{\tau}(t) \leq C, \quad 1 \leq d \leq 3,$$

where, here and in the remainder of the proof, the constant C depends only on q, $\max_x |u_0(x)|$, the spatial dimension $1 \leq d \leq 3$, and T but may change its value from one line to the next. Remembering that $\delta_n \searrow 0$, the independence in l is trivially seen since Lemma 2.2 implies

$$\sum_{y \in \mathbb{X}_{n,l}^d} \left[\mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right]^2 \leq \sum_{y \in \mathbb{X}_n^d} \left[\mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right]^2 \leq \frac{C}{t^{d/4}}; \forall 1 \leq d \leq 3, l \in \mathbb{N}$$

Similarly, letting $X_{n,l,i,R}^{x,\tau}$ denote the random part of $X_{n,l,i}^{x,\tau}$ on the truncated lattice $\mathbb{X}_{n,l}^d$, using (2.54), and repeating the arguments in Lemma 2.5 and Lemma 2.6—replacing $\mathbb{X}_n^d$ by $\mathbb{X}_{n,l}^d$ and noting that Lemma 2.3 and Lemma 2.4 hold on $\mathbb{X}_{n,l}^d$—we obtain

$$
\begin{aligned}
& \mathbb{E} \left| X_{n,l,i,R}^{x,\tau_1}(t) - X_{n,l,i,R}^{y,\tau_1}(t) \right|^{2q} + \mathbb{E} \left| X_{n,l,i,R}^{x,\tau_2}(t) - X_{n,l,i,R}^{y,\tau_2}(t) \right|^{2q} \\
& \leq C_d |x-y|^{2q\alpha_d}; \ \alpha_d \in I_d, \\
& \mathbb{E} \left| X_{n,l,i,R}^{x,\tau_1}(t) - X_{n,l,i,R}^{x,\tau_1}(r) \right|^{2q} + \mathbb{E} \left| X_{n,l,i,R}^{x,\tau_2}(t) - X_{n,l,i,R}^{x,\tau_2}(r) \right|^{2q} \\
& \leq C \left| t-r \right|^{\frac{(4-d)q}{4}},
\end{aligned}
$$

(2.55)

for all $x, y \in \mathbb{X}_{n,l}^d$, $r, t \in [0, \tau_1 \wedge \tau_2]$, $\tau_1, \tau_2 \in \mathbb{T}$, and $1 \leq d \leq 3$. It follows that, for every point $\tau_{1,2} = (\tau_1, \tau_2) \in \mathbb{T}^2$, there is a subsequence $\left\{ \left(\tilde{X}_{n,l,i_m}^{\tau_1}, \tilde{X}_{n,l,i_m}^{\tau_2} \right) \right\}_{m=1}^{\infty}$ on a probability space $(\tilde{\Omega}_{\tau_{1,2}}, \tilde{\mathscr{F}}_{\tau_{1,2}}, \tilde{\mathbb{P}}_{\tau_{1,2}})$ such that $\left(\tilde{X}_{n,l,i_m}^{\tau_1}, \tilde{X}_{n,l,i_m}^{\tau_2} \right) \stackrel{\mathscr{L}}{=} \left(X_{n,l,i_m}^{\tau_1}, X_{n,l,i_m}^{\tau_2} \right)$ and $\left(\tilde{X}_{n,l,i_m}^{x,\tau_1}(t), \tilde{X}_{n,l,i_m}^{x,\tau_2}(t) \right) \longrightarrow \left(\tilde{X}_{n,l}^{x,\tau_1}(t), \tilde{X}_{n,l}^{x,\tau_2}(t) \right)$ uniformly on compact subsets of $[0, \tau_1 \wedge \tau_2] \times \mathbb{X}_n^d$, as $m \to \infty$ a.s. Let $\mathbb{T}_{\mathbb{Q}} = \mathbb{T} \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals, and define the product probability space

$$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}) := \left(\bigotimes_{\tau_{1,2} \in \mathbb{T}_{\mathbb{Q}}^2} \tilde{\Omega}_{\tau_{1,2}}, \bigotimes_{\tau_{1,2} \in \mathbb{T}_{\mathbb{Q}}^2} \tilde{\mathscr{F}}_{\tau_{1,2}}, \bigotimes_{\tau_{1,2} \in \mathbb{T}_{\mathbb{Q}}^2} \tilde{\mathbb{P}}_{\tau_{1,2}} \right).$$

If $s < t$, then for every $f \in C_b^2(\mathbb{R}^2; \mathbb{R})$, $\tau_1, \tau_2 \in \mathbb{T}_{\mathbb{Q}} \setminus \{0\}$, $t \in [0, \tau_1 \wedge \tau_2]$, $x_1, x_2 \in \mathbb{X}_{n,l}^d$, and for every bounded continuous $F : C\left(\mathbb{R}_+; \mathbb{R}^2\right) \to \mathbb{R}$ that

is $\mathscr{B}_s\left(\mathrm{C}\left(\mathbb{R}_+; \mathbb{R}^2\right)\right) := \sigma\left(z(r); 0 \le r \le s\right)$-measurable function, we have

$$
\begin{aligned}
(2.56) \quad & \mathbb{E}_{\tilde{\mathbb{P}}}\left[\left\{f(\tilde{Z}_{n,l}^{x_{1,2},\tau_{1,2}}(t)) - f(\tilde{Z}_{n,l}^{x_{1,2},\tau_{1,2}}(s))\right.\right. \\
& \left.- \int_s^t \left(\mathscr{A}_\Upsilon^{\tau_{1,2}} f\right)(r, x_1, x_2, \tilde{Z}_{n,l}^{\tau_{1,2}})dr\right\} F\left(\tilde{Z}_{n,l}^{x_{1,2},\tau_{1,2}}(\cdot)\right)\Big] \\
& = \lim_{m \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\left\{f(\tilde{Z}_{n,l,i_m}^{x_{1,2},\tau_{1,2}}(t)) - f(\tilde{Z}_{n,l,i_m}^{x_{1,2},\tau_{1,2}}(s))\right.\right. \\
& \left.- \int_s^t \left(\mathscr{A}_{\Upsilon,i_m}^{\tau_{1,2}} f\right)(r, x_1, x_2, \tilde{Z}_{n,l,i_m}^{\tau_{1,2}})dr\right\} F\left(\tilde{Z}_{n,l,i_m}^{x_{1,2},\tau_{1,2}}(\cdot)\right)\Big] = 0,
\end{aligned}
$$

where, by a standard localization argument, we have assumed that a is also bounded; and where $\tilde{Z}_{n,l}^{\tau_{1,2}}$ and $\tilde{Z}_{n,l,i_m}^{\tau_{1,2}}$ are obtained from the definition of $Z_{n,l}^{\tau_{1,2}}$ in (1.19) by replacing $X_{n,l}^{\tau_j}$ by $\tilde{X}_{n,l}^{\tau_j}$ and $\tilde{X}_{n,l,i_m}^{\tau_j}$, $j = 1, 2$, respectively. The operator $\mathscr{A}_{\Upsilon,i_m}^{\tau_{1,2}}$ is obtained from $\mathscr{A}_\Upsilon^{\tau_{1,2}}$ by replacing $\Upsilon_{\delta_n,t,\tau_{i,j}}^{x_{i,j},y}\left(u^y(t)\right)$ in (1.18) by $\Upsilon_{\delta_n,\phi_{i_m}(t),\tau_{i,j}}^{x_{i,j},y}\left(u^y(\phi_{i_m}(t))\right)$. Also, obviously, for any $\tau \in \mathbb{T}_\mathbb{Q}$ and $t \in [0, \tau]$

$$(2.57) \quad \tilde{X}_{n,l}^{x,\tau}(t) = \lim_{m \to \infty} \tilde{X}_{n,l,i_m}^{x,\tau}(t) = \tilde{U}_{n,D}^x(t); \qquad x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d, \text{ a.s. } \tilde{\mathbb{P}}.$$

It follows from (2.56) and (2.57) that $\left\{\tilde{X}_{n,l}^\tau\right\}_{\tau \in \mathbb{T}_\mathbb{Q}}$ satisfies the K-martingale problem (KM) with respect to the filtration $\{\tilde{\mathscr{F}}_t\}$, with

$$\tilde{\mathscr{F}}_t = \bigcap_{\epsilon > 0} \sigma\left\{\tilde{X}_{n,l}^{x,\tau}(u); u \le (t + \epsilon) \wedge \tau, \tau \in \mathbb{T}_\mathbb{Q} \cap (t, T]\right\}.$$

Thus, by Theorem 1.1, with $\tau \in \mathbb{R}_+$ replaced by $\tau \in \mathbb{T}_\mathbb{Q}$, there is a solution $\tilde{U}_{n,l}^x(t)$ to the l-truncated BTRW SIE (1.13) on $\mathbb{T}_\mathbb{Q} \times \mathbb{X}_n^d$. Use continuous extension in time of $\tilde{U}_{n,l}^x(t)$ to extend its definition to $\mathbb{T} \times \mathbb{X}_n^d$, and denote the extension also by $\tilde{U}_{n,l}^x(t)$. Clearly $\tilde{U}_{n,l}^x(t)$ solves the l-truncated BTRW SIE (1.13) on $\mathbb{T} \times \mathbb{X}_n^d$.

Now, for $q \ge 1$, let $M_{q,l}(t) = \sup_{x \in \mathbb{X}_n^d} \mathbb{E}\left|\tilde{U}_{n,l}^x(t)\right|^{2q}$. As above, the boundedness of $\tilde{U}_{n,D}^x(t)$, implies

$$(2.58) \quad M_{q,l}(t) \le C + \sup_{x \in \mathbb{X}_{n,l}^d} \mathbb{E}\left|\tilde{U}_{n,l}^x(t)\right|^{2q}.$$

Then, replacing $\mathbb{X}_n^d$ by $\mathbb{X}_{n,l}^d$ and following the same steps as in the proof of Proposition 2.1, we get that

$$(2.59) \quad M_{q,l}(t) \le C, \quad \forall t \in \mathbb{T} \text{ and } 1 \le d \le 3.$$

Similarly, letting $\tilde{U}_{n,l,R}^x(t)$ denote the random part of $\tilde{U}_{n,l}^x(t)$ on the truncated lattice $\mathbb{X}_{n,l}^d$, using (2.59), and repeating the arguments in Lemma 2.5 and Lemma 2.6—replacing $\mathbb{X}_n^d$ by $\mathbb{X}_{n,l}^d$ and noting that the inequalities in Lemma 2.3 and Lemma 2.4 trivially hold if we replace $\mathbb{X}_n^d$ by $\mathbb{X}_{n,l}^d$—we

obtain

(2.60)
$$\mathbb{E}\left|\tilde{U}_{n,l,R}^x(t) - \tilde{U}_{n,l,R}^y(t)\right|^{2q} \leq C_d|x-y|^{2q\alpha_d}; \ \alpha_d \in I_d,$$

$$\mathbb{E}\left|\tilde{U}_{n,l,R}^x(t) - \tilde{U}_{n,l,R}^x(r)\right|^{2q} \leq C\left|t-r\right|^{\frac{(4-d)q}{4}},$$

for all $x, y \in \mathbb{X}_{n,l}^d$, $r, t \in \mathbb{T}$, and $1 \leq d \leq 3$. By Remark 1.2, $\tilde{U}_{n,D}^x(t)$ is differentiable in t. So, linearly interpolating $\tilde{U}_{n,l}^x(t)$ in space and using (2.60) and arguing as in the proof of part (a) of Lemma 2.8, we get that the continuous map $(t,x) \mapsto \tilde{U}_{n,l}^x(t)$ is locally γ_t-Hölder continuous in time with $\gamma_t \in \left(0, \frac{4-d}{8}\right)$ for $1 \leq d \leq 3$.

(b) Clearly, $\tilde{U}_{n,D}^x(t)$ in (1.13) is the same for every l, so it is enough to show convergence of the random part $\tilde{U}_{n,l,R}^x(t)$. Using (2.60) we get tightness for $\left\{\tilde{U}_{n,l,R}^x(t)\right\}_l$ and consequently a subsequential weak limit $\tilde{U}_n$, which is our limit solution for $e_{\mathrm{BTRW}}^{\mathrm{SIE}}(a, u_0, n)$. For the regularity assertion, $\tilde{U}_{n,D}^x(t)$ is smooth and bounded as noted above. So, using (2.59) and (2.60), and imitating the argument in the proof of part (b) of Lemma 2.8 (remembering that here we are taking the limit as $l \to \infty$); we get the desired L^p boundedness for $\tilde{U}_n$ as in Proposition 2.1 and the spatial and temporal moments bounds in Lemma 2.5 and Lemma 2.6

(2.61)
$$\begin{cases} \mathbb{E}\left|\tilde{U}_n^x(t)\right|^{2q} \leq C \\ \mathbb{E}\left|\tilde{U}_{n,R}^x(t) - \tilde{U}_{n,R}^y(t)\right|^{2q} \leq C_d|x-y|^{2q\alpha_d}; \ \alpha_d \in I_d, \\ \mathbb{E}\left|\tilde{U}_{n,R}^x(t) - \tilde{U}_{n,R}^x(r)\right|^{2q} \leq C\left|t-r\right|^{\frac{(4-d)q}{4}}, \end{cases}$$

for $(t, x, n) \in \mathbb{T} \times \mathbb{X}_n^d \times \mathbb{N}$ and for $d = 1, 2, 3$ and $q \geq 1$ and the desired Hölder regularity follows.

The proof is complete. □

We now get part (b) of Theorem 1.2 for $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ as the following

Corollary 2.3. *Theorem 1.2 (b) holds.*

Proof. The desired conclusion follows upon using the argument in the proof of part (b) of Lemma 2.8 along with Definition 1.4 and the L^p-boundedness and the spatial and temporal moments bounds for $\left\{\tilde{U}_n\right\}_n$ that we got in (2.61) above. □

2.5. **BTP SIEs-fourth order parametrized BTP SPDEs link.** The proof of Lemma 1.2 can be easily handled using an applicationt of Itô's formula. We give a simple derivation below.

Proof of Lemma 1.2. Fix an arbitrary $(n, t, x) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{X}_n^d$. For any $l \in \mathbb{N}$, let $\mathbb{X}_{n,l,x}^d$ be the finite sublattice of $\mathbb{X}_n^d$ centered around x and of radius l; i.e., $\mathbb{X}_{n,l,x}^d = \mathbb{X}_n^d \cap \otimes_{i=1}^d [x_i - l, x_i + l]$ (x_i is the i-th coordinate of x) and let $r = \# \left\{y; y \in \mathbb{X}_{n,l,x}^d\right\}$

and $\left\{ y; y \in \mathbb{X}_{n,l,x}^d \right\} = \left\{ y^{(1)}, \ldots, y^{(r)} \right\}$. Let $\tilde{U}_n$ be a continuous (in s) solution to $e_{\mathrm{BTRW}}^{\mathrm{PSDDE}}(a, u_0, n)$; i.e., the semimartingale (in s) satisfying (1.24). Let

$$F_l(t - s, x, \tilde{U}_n^{x,\cdot}(s,t)) := \sum_{y \in \mathbb{X}_{n,l,x}^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \tilde{U}_n^{x,y}(s,t) = \sum_{k=1}^r \mathbb{K}_{t-s;x,y^{(k)}}^{\mathrm{BTRW}_{\delta_n}^d} \tilde{U}_n^{x,y^{(k)}}(s,t)$$

We denote by $\partial_i F_l$ the first derivative of F_l in the i-th variable, $i = 1, 2$, with $\nabla_3 F_l$ being the r-dimensional gradient vector of F_l in the third argument $\tilde{U}_n$ at all $y^{(1)}, \ldots, y^{(r)} \in \mathbb{X}_{n,l,x}^d$; i.e., formally,

$$\nabla_3 F_l(t - s, x, \tilde{U}_n^{x,\cdot}(s,t)) := \left(\frac{\partial F_l}{\partial \tilde{U}_n^{x,y^{(1)}}(s,t)}, \ldots, \frac{\partial F_l}{\partial \tilde{U}_n^{x,y^{(r)}}(s,t)} \right).$$

Applying Itô's formula to F_l, and remembering that $X_s^{(1)} = t - s$ is of bounded variation with $dX_s^{(1)} = -ds$, $X_s^{(2)} = x$ and $dX_s^{(2)} = 0$, $X_s^{(k)} = \tilde{U}_n^{x,y^{(k-2)}}(s,t)$ for $k = 3, \ldots, 2 + r$, $\{W_n^y(s)\}_{y \in \mathbb{X}_n^d}$ is a collection of independent BMs, all second partials for $k = 3, \ldots, 2 + r$ are zero at every $y \in \mathbb{X}_{n,l,x}^d$—since F_l is linear in $\left\{ \tilde{U}_n^{x,y^{(k-2)}}(s,t) \right\}_{k=3}^{2+r}$, and all the differentials below are in s, we get

$$\tilde{U}_n^x(t) - \sum_{y \in \mathbb{X}_{n,l,x}^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) = F_l(0, x, \tilde{U}_n^{x,\cdot}(t,t)) - F_l(t, x, u_0)$$

$$= -\int_0^t \partial_1 F_l(t - s, x, \tilde{U}_n^{x,\cdot}(s,t)) ds + \int_0^t \left(\nabla_3 F_l(t - s, x, \tilde{U}_n^{x,\cdot}(s,t)), d\tilde{U}_n^{x,\cdot}(s,t) \right)$$

$$= -\sum_{y \in \mathbb{X}_{n,l,x}^d} \int_0^t \left[\frac{\partial}{\partial(t - s)} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right] \tilde{U}_n^{x,y}(s,t) ds + \sum_{y \in \mathbb{X}_{n,l,x}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} d\tilde{U}_n^{x,y}(s,t)$$

Using Lemma 1.1 and (1.27), we see the above is

$$= -\sum_{y \in \mathbb{X}_{n,l,x}^d} \int_0^t \left[\frac{\Delta_n \mathbb{K}_{0;x,y}^{\mathrm{BTRW}_{\delta_n}^d}}{\sqrt{8\pi(t - s)}} + \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right] \tilde{U}_n^{x,y}(s,t) ds$$

$$+ \sum_{y \in \mathbb{X}_{n,l,x}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[\frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t - s)}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^{x,y}(s,t) \right] ds$$

$$+ \sum_{y \in \mathbb{X}_{n,l,x}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right]$$

Taking the limit as $l \to \infty$, using the conditions on $\tilde{U}$, we get

$$\tilde{U}_n^x(t) - \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) = \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right]$$

$$- \sum_{y \in \mathbb{X}_n^d} \int_0^t \left[\frac{\Delta_n \mathbb{K}_{0;x,y}^{\mathrm{BTRW}_{\delta_n}^d}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right] \tilde{U}_n^{x,y}(s,t) ds$$

$$+ \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[\frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^{x,y}(s,t) \right] ds$$

Now, remember that $\sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \equiv 1$ for every $(s,t,x) \in [0,t] \times \mathbb{R}_+ \times \mathbb{X}_n^d$, that the term $\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x} / \sqrt{8\pi(t-s)}$ does not depend on y, and use summation by parts four times to get

$$\tilde{U}_n^x(t) - \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) = \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}}$$

$$- \int_0^t \frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} ds - \sum_{y \in \mathbb{X}_n^d} \int_0^t \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \tilde{U}_n^{x,y}(s,t) ds + \int_0^t \frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} ds$$

$$+ \sum_{y \in \mathbb{X}_n^d} \int_0^t \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \tilde{U}_n^{x,y}(s,t) ds = \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}},$$

which is what we wanted to show. $\square$

3. Conclusions

We considered the multiplicative noise case for our recently introduced (see [1]) fourth order BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ in (1.3) on $\mathbb{R}_+ \times \mathbb{R}^d$, under both Lipschitz and less than Lipschitz conditions on the diffusion coefficient a. We analyzed our BTP SIE using a numerically-flavored lattice approach similar to our discretized method for the simpler second order RD SPDEs driven by space-time white noise (see [11, 7]). We discretize space and formulate solutions to the resulting spatially-discrete stochastic integral equations in terms of the density of Brownian-time random walk or BTRW—which we introduce in this article along with the general class of Brownian-time chains (BTCs), of which BTRW is a special case. As with their continuous counterpart BTPs, which we introduced in [9, 8], BTCs are interesting new processes outside of the current well established theory; and we believe they merit further study. In the course of proving our results, we prove several interesting facts about the BTRW. These include a connection to fourth order differential-difference equation that is proved in Appendix A and different estimates which lead to the definition of 2-Brownian-times Brownian motion and 2-Brownian-times random walk. We define two notions of solutions to the lattice model: direct solutions and limit solutions (from a finite truncation of the lattice to the whole lattice). These solutions (both direct and limit) are then used to define two types

of BTRW SIEs limit solutions to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ (direct limit solutions and double limit solutions), as the size of the lattice mesh shrinks to zero.

The densities of BTRW and BTBM have a considerable regularizing effect on stochastic kernel equations driven by space-time white noise as compared to the standard Brownian motion or continuous time random walk densities (the usual green functions for second order RD equations and their spatially-discretized versions). The unconventional memory-preserving fourth order PDEs associated with the BTP density are highly regular: their solutions are $\mathrm{C}^{1,4}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ for all times and all $d \geq 1$, despite the fact that a positive bi-Laplacian term is part of the equation [9, 8]. However, this bi-Laplacian is coupled in a very specific way—dictated by the BTP probability density function—with a Laplacian acting on the smooth initial data and whose coefficient grows arbitrarily large as time approaches the initial time (zero) at the rate of $1/\sqrt{8\pi t}$. One way to understand this smoothing effect is to note that the BTP density in $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ is intimately connected—and shares regularity properties—with the kernel associated with our recently introduced imaginary-Brownian-time-Brownian-angle process, which we used to give a solution to a Kuramoto-Sivashisky-type PDE in [2]. In the stochastic setting of this article, this BTP density smoothing effect on $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ is evidenced in a regularity of solutions that is much higher than typical second order space-time white noise driven RD SPDEs. This regularizing effect is such that we are able to obtain γ-Hölder continuous solutions to our BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ for spatial dimensions $1 \leq d \leq 3$. We show that the Hölder exponent is dimension-dependent with $\gamma \in (0, \frac{4-d}{8})$, $1 \leq d \leq 3$. In addition, we are able to show ultra spatial regularity by showing a remarkable nearly local Lipschitz regularity for $d = 1, 2$, and nearly local Hölder $1/2$ regularity in $d = 3$. This gives, for the first time, an example of a kernel—that is also the density of an interesting stochastic process—that is able to regularize a space-time white noise driven equation so that its solutions are pushed beyond the traditional nearly Hölder $1/2$ spatial regularity. Of course this contrasts sharply with second order RD SPDEs driven by space-time white noise whose counterpart real-valued solutions are confined to the case $d = 1$. This contrast is summarized in Table 3.1 below. For the precise regularity results please

Spatial Dimension	Real-Valued Solution		Hölder Exponent (time, space)	
	RD SPDEs	BTP SIEs	RD SPDEs	BTP SIEs
$d = 1$	Yes	Yes	$\left(\frac{1}{4}^-, \frac{1}{2}^-\right)$	$\left(\frac{3}{8}^-, 1^-\right)$
$d = 2$	No	Yes	N/A	$\left(\frac{1}{4}^-, 1^-\right)$
$d = 3$	No	Yes	N/A	$\left(\frac{1}{8}^-, \frac{1}{2}^-\right)$

TABLE 3.1. BTP SIEs vs RD SPDEs

consult Theorem 1.2 and Remark 1.4. To get the smoothing effect of the BTP density, we prove BTRW estimates that enable us to extract a BTRW SIEs weak limit (direct limit in the Lipschitz case and double limits in the non-Lipschitz one) Hölder continuous solution to $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$ in spatial dimensions $d = 1, 2, 3$, in spite of the presence of the driving space-time white noise. Again, the Hölder exponent depends on the space dimension through the expression $(4 - d)/8$. This ultra regularity in multispatial dimensions naturally motivates the study of the variations

(temporal and spatial) of BTP SIEs, which we undertake—among other aspects of BTP SIEs—in [5] and a followup article.

Encoding this smoothing effect from our BTP SIE $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$ into a fourth order SPDE involving the bi-Laplacian coupled with a Laplacian term requires extra parameters. We give what we call the fourth order parametrized SPDE corresponding to $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$, linking the spatially-discretized BTRW SIE to the diagonals of a parametrized stochastic differential-difference equation (BTRW PSDDE) on the lattice.

To deal with the non-Lipschitz condition on a, we introduce our K-martingale approach, which is tailor-made for kernel SIEs as $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$ as well as for other mild formulations of many other different SPDEs. It is a delicate variant of the well known, and by now classic, martingale problem approach of Stroock and Varadhan for SDEs. A key advantage of the K-martingale approach is that it is a unified framework in which the existence and uniqueness of many kernel stochastic integral equations, which are the mild formulation for many SPDEs, may be treated using only variants of the kernel SIE. This includes SPDEs of different orders (second and fourth), so long as the corresponding spatially-discretized kernel (or density) satisfies Kolmogorov-type bounds on its temporal and spatial differences. In essence, what the K-martingale approach implies is that if the kernel in the lattice model is nice enough for the lattice model to converge as the lattice mesh shrinks to zero (under appropriate assumptions on a), then it is nice enough to guarantee a solution for the lattice model. We use it here to prove the existence of BTRW SIEs double limit solutions to (1.3) under the conditions (NLip), but just as with the Stroock-Varadhan method, it can handle uniqueness as well. Our K-martingale approach starts by constructing an auxiliary problem to a truncated lattice version of (1.3), for which the existence of solutions implies solutions existence for the truncated lattice model. We then formulate a martingale problem equivalent to the auxilliary problem (the K-martingale problem).

As we recently started doing for PDEs [2], we adapt the methods presented here and in [2] to give an entirely new approach—in terms of our Linearized Kuramoto-Sivashinsky process (or imaginary-Brownian-time-Brownian-angle process) and related processes—to study the multi-spatial dimensions SPDEs version of famous fourth order applied mathematics PDEs like the Kuramoto-Sivashinsky (several different versions), the Cahn-Hilliard, and the Swift-Hohenberg PDEs. We illustrate this in upcoming papers ([3, 4, 6]) and planned followup papers. Traditional semigroup analytical methods alone are not adequate for this since the existence of the KS semigroup in $d > 1$ is not settled analytically.

Also, SIEs corresponding to other BTP processes we introduced in [8] may also be studied by adapting and generalizing our approach here. We believe BTPs, their PDEs, their SIEs, and their discretized cousins (the BTCs and their equations) can play a useful role by adding new, currently unavailable, insights and models to the ever growing mathematical finance theory. We also hope to explore these aspects in future papers.

work as well as parts of the parallel work on Kuramoto-Sivashinsky-type SPDEs. My thanks are also to José Langa, Yimin Xiao, and Jinqiao Duan for stimulating discussions on related ongoing collaborative work. Finally, I thank Dan Stroock for his input on Harmonic analysis aspects of BTBM.

APPENDIX A. PROOF OF BTRW-DDE CONNECTION

In this appendix, we give the proof of Lemma 1.1 linking the density of BTRW to fourth order differential-difference equations.

Proof of Lemma 1.1. Let $u_n^x(t) = \mathbb{E}\left[u_0\left(\mathbb{S}_{B,\delta_n}^x(t)\right)\right]$ with u_0 as in (NLip). Observe that

$$(A.1) \qquad \mathbb{E}\left[u_0\left(\mathbb{S}_{B,\delta_n}^x(t)\right)\right] = 2\int_0^\infty K_{t;0,s}^{\mathrm{BM}} \mathbb{E}\left[u_0\left(S_{\delta_n}^x(s)\right)\right] ds$$

where $K_{t;0,s}^{\mathrm{BM}}$ the transition density of the one dimensional BM $B(t)$. Differentiating (A.1) with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $\partial K_{t;0,s}^{\mathrm{BM}}/\partial t = \frac{1}{2}\partial^2 K_{t;0,s}^{\mathrm{BM}}/\partial s^2$ we have

$$
(A.2) \qquad
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left[u_0\left(\mathbb{S}_{B,\delta_n}^x(t)\right)\right] &= 2\int_0^\infty \frac{\partial}{\partial t}K_{t;0,s}^{\mathrm{BM}}\mathbb{E}\left[u_0\left(S_{\delta_n}^x(s)\right)\right] ds \\
&= \int_0^\infty \frac{\partial^2}{\partial s^2}K_{t;0,s}^{\mathrm{BM}}\mathbb{E}\left[u_0\left(S_{\delta_n}^x(s)\right)\right] ds.
\end{aligned}
$$

Letting $\mathscr{T}_{n,s}u_0(x) = \mathbb{E}u_0(S_{\delta_n}^x(s))$ be the action of the semigroup $\mathscr{T}_{n,s}$ associated with the standard continuous-time symmetric random walk $S_{\delta_n}^x$ on the lattice $\mathbb{X}_n^d$. Then, the generator of $S_{\delta_n}^x$ on $\mathbb{X}_n^d$ is given by $\mathscr{A}_n = \Delta_n/2$. Alternatively, noting that $K_{t;x}^{\mathrm{RW}_{\delta_n}^d}$ is the fundamental solution to the deterministic heat equation (1.8) on the lattice $\mathbb{X}_n^d$, we get

$$
\begin{aligned}
\frac{d}{ds}\mathscr{T}_{n,s}u_0(x) &= \frac{d}{ds}\mathbb{E}\left[u_0\left(S_{\delta_n}^x(s)\right)\right] = \sum_{y\in\mathbb{X}_n^d} u_0(y)\frac{d}{ds}K_{s;x,y}^{\mathrm{RW}_{\delta_n}^d} \\
&= \frac{1}{2}\sum_{y\in\mathbb{X}_n^d} u_0(y)\Delta_n K_{s;x,y}^{\mathrm{RW}_{\delta_n}^d} = \frac{1}{2}\Delta_n \sum_{y\in\mathbb{X}_n^d} u_0(y)K_{s;x,y}^{\mathrm{RW}_{\delta_n}^d} \\
&= \frac{1}{2}\Delta_n\mathbb{E}\left[u_0\left(S_{\delta_n}^x(s)\right)\right] = \mathscr{A}_n\mathscr{T}_{n,s}u_0(x).
\end{aligned}
$$

So, we integrate (A.2) by parts twice, and we observe that the boundary terms always vanish at ∞ (as $s \nearrow \infty$) and that we have $(\partial/\partial s)K_{t;0,s}^{\mathrm{BM}} = 0$ at $s = 0$ but $K_{t;0,0}^{\mathrm{BM}} > 0$. This gives us

$$
(A.3) \qquad
\begin{aligned}
\frac{d}{dt}u_n^x(t) = \frac{d}{dt}\mathbb{E}\left[u_0\left(\mathbb{S}_{B,\delta_n}^x(t)\right)\right] &= -\int_0^\infty \frac{\partial}{\partial s}K_{t;0,s}^{\mathrm{BM}}\frac{d}{ds}\mathscr{T}_{n,s}u_0(x)ds \\
&= K_{t;0,0}^{\mathrm{BM}}\mathscr{A}_n u_0(x) + \int_0^\infty K_{t;0,s}^{\mathrm{BM}}\mathscr{A}_n^2\mathscr{T}_{n,s}u_0(x)ds \\
&= K_{t;0,0}^{\mathrm{BM}}\mathscr{A}_n u_0(x) + \mathscr{A}_n^2\int_0^\infty K_{t;0,s}^{\mathrm{BM}}\mathscr{T}_{n,s}u_0(x)ds \\
&= \frac{\Delta_n u_0(x)}{\sqrt{8\pi t}} + \frac{1}{8}\Delta_n^2 u_n^x(t).
\end{aligned}
$$

Obviously, $u_n^x(0) = u_0(x)$, and we have proven (1.9).

Of course, if $u_n^x(t) = \mathbb{K}_{t,x}^{\text{BTRW}_{\delta_n}^d}$, then $u_0(x) = \mathbb{K}_{0;x}^{\text{BTRW}_{\delta_n}^d}$ as given in Lemma 1.1, and we have by (1.7) and by the steps above

$$\frac{d}{dt}u_n^x(t) = \frac{d}{dt}\mathbb{K}_{t,x}^{\text{BTRW}_{\delta_n}^d} = 2\int_0^\infty \frac{\partial}{\partial t}K_{t;0,s}^{\text{BM}}K_{s;x}^{\text{RW}_{\delta_n}^d}\,ds = \int_0^\infty \frac{\partial^2}{\partial s^2}K_{t;0,s}^{\text{BM}}K_{s;x}^{\text{RW}_{\delta_n}^d}\,ds$$

$$= \frac{\Delta_n K_{0;x}^{\text{RW}_{\delta_n}^d}}{\sqrt{8\pi t}} + \frac{1}{8}\Delta_n^2 u_n^x(t) = \frac{\Delta_n \mathbb{K}_{0;x}^{\text{BTRW}_{\delta_n}^d}}{\sqrt{8\pi t}} + \frac{1}{8}\Delta_n^2 u_n^x(t).$$

The proof is complete. $\square$

Appendix B. The Lipschitz case directly

We now state and directly (without discretization) prove strong existence and uniqueness for our BTP SIE on $\mathbb{R}_+ \times \mathbb{R}^d$ under Lipschitz conditions.

Theorem B.1 (Existence, uniqueness, and regularity of direct solutions for dimensions $1 \leq d \leq 3$). *Fix an arbitrary $T > 0$, and let $\mathbb{T} = [0, T]$. Assume that* (Lip) *holds. Then there exists a pathwise- unique strong solution $(U, \mathscr{W})$ to $e_{BTP}^{SIE}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$, for $1 \leq d \leq 3$, which is $L^p(\Omega)$-bounded on $\mathbb{T} \times \mathbb{R}^d$ for all $p \geq 2$. Furthermore, $U \in \mathrm{H}^{\frac{4-d}{8}, \alpha_d}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $1 \leq d \leq 3$ and every $\alpha_d \in I_d$, where the intervals I_d are given in Theorem 1.2.*

Before we prove Theorem B.1, we start by recalling a useful elementary Gronwall-type lemma whose proof can be found in Walsh [35].

Lemma B.1. *Let $\{g_n(t)\}_{n=0}^\infty$ be a sequence of positive functions such that g_0 is bounded on $\mathbb{T} = [0, T]$ and*

$$g_n(t) \leq C\int_0^t g_{n-1}(s)(t-s)^\alpha ds, \quad n = 1, 2, \ldots$$

for some constants $C > 0$ and $\alpha > -1$. Then, there exists a (possibly different) constant $C > 0$ and an integer $k > 1$ such that for each $n \geq 1$ and $t \in \mathbb{T}$

$$g_{n+mk}(t) \leq C^m \int_0^t g_n(s)\frac{t-s}{(m-1)!}ds; \quad m = 1, 2, \ldots.$$

We are now ready for the

Proof of Theorem B.1.   For the existence proof, we construct a solution iteratively. So, given a space-time white noise $\mathscr{W}$, on some $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$, define

(B.1)
$$\begin{cases} U^{(0)}(t, x) = \displaystyle\int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BTBM}^d} u_0(y)dy \\ U^{(n+1)}(t, x) = U^{(0)}(t, x) + \displaystyle\int_{\mathbb{R}^d}\int_0^t \mathbb{K}_{t-s;x,y}^{\text{BTBM}^d} a(U^{(n)}(s, y))\mathscr{W}(ds \times dy) \end{cases}$$

We will show that, for any $p \geq 2$ and all $1 \leq d \leq 3$, the sequence $\left\{U^{(n)}(t, x)\right\}_{n \geq 1}$ converges in $L^p(\Omega)$ to a solution. Let

$$D_{n,p}(t, x) := \mathbb{E}\left|U^{(n+1)}(t, x) - U^{(n)}(t, x)\right|^p$$

$$D_{n,p}^*(t) := \sup_{x \in \mathbb{R}^d} D_{n,p}(t,x).$$

Starting with the case $p > 2$, we bound $D_{n,p}$ using Burkholder inequality, the Lipschitz condition (a) in (Lip), and then Hölder inequality with $0 \leq \epsilon \leq 1$ and $q = p/(p-2)$ to get

$$D_{n,p}(t,x) = \mathbb{E}\left| \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} \left[a(U^{(n)}(s,y)) - a(U^{(n-1)}(s,y)) \right] \mathscr{W}(ds \times dy) \right|^p$$

$$\leq C\mathbb{E}\left| \int_{\mathbb{R}^d} \int_0^t \left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} \right)^2 \left[U^{(n)}(s,y) - U^{(n-1)}(s,y) \right]^2 dsdy \right|^{p/2}$$

$$\leq C \left(\int_{\mathbb{R}^d} \int_0^t \left[\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} \right]^{2\epsilon q} dsdy \right)^{p/2q}$$

$$\times \int_{\mathbb{R}^d} \int_0^t \left(\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} \right)^{(1-\epsilon)p} D_{n-1,p}(s,y)dsdy$$

Take $\epsilon = (p-2)/p$ in the above ($2\epsilon q = (1-\epsilon)p = 2$), take the supremum over the space variables, and use the computation on p. 531 of [1] to see that, for $1 \leq d \leq 3$ the above reduces to

(B.2) $$D_{n,p}^*(t) \leq C \left(t^{\frac{4-d}{4}} \right)^{\frac{p-2}{2}} \int_0^t D_{n-1,p}^*(s) \left[t - s \right]^{-\frac{d}{4}} ds$$

The case $p = 2$ is simpler. We apply Burkholder's inequality to $D_{n,2}$ and then take the space supremum to get

(B.3) $$D_{n,2}^*(t) \leq C \int_0^t D_{n-1,2}^*(s) \left[t - s \right]^{-\frac{d}{4}} ds$$

I.e., on any time interval $\mathbb{T} = [0, T]$, the integral multiplier on the r.h.s. of (B.2) is bounded; and if $D_{n-1,p}^*$ is bounded on $\mathbb{T}$ then so is $D_{n,p}^*$, for every $p \geq 2$. Now,

$$D_{0,p}^*(t) \leq C \sup_{x \in \mathbb{R}^d} \mathbb{E}\left| \int_{\mathbb{R}^d} \int_0^t \left[\mathbb{K}_{t-s;x,y}^{\mathrm{BTBM}^d} \right]^2 a^2 \left(U^{(0)}(s,y) \right) \right|^{\frac{p}{2}}$$

Since u_0 is bounded and deterministic, then so are $U^{(0)}$ and $a(U^{(0)})$. The latter assertion follows from the growth condition on a in (Lip). Thus, by the computation on p. 531 of [1], $D_{0,p}^*$ is bounded on $\mathbb{T}$ for $1 \leq d \leq 3$ and so are all the $D_{n,p}^*$. Lemma B.1 now implies that for each $1 \leq d \leq 3$, the series $\sum_{m=0}^\infty \left[D_{n+mk,p}^*(t) \right]^{1/p}$ converges uniformly on compacts for each n, which in turn implies that $\sum_{n=0}^\infty \left[D_{n,p}^*(t) \right]^{1/p}$ converges uniformly on compacts. Thus $U^{(n)}$ converges in $L^p(\Omega)$ for $p \geq 2$, uniformly on $\mathbb{T} \times \mathbb{R}^d$ for $1 \leq d \leq 3$. Let $U(t,x) := \lim_{n \to \infty} U^{(n)}(t,x)$. It is easy to see that U satisfies (1.3), and hence solves the BTP SIE $e_{\mathrm{BTP}}^{\mathrm{SIE}}(a, u_0)$. This follows from (B.1) since the Lipschitz condition in (Lip) gives

$$\mathbb{E}\left| a(U(t,x)) - a(U^{(n)}(t,x)) \right|^2 \leq C\mathbb{E}\left| U(t,x) - U^{(n)}(t,x) \right|^2 \to 0 \quad \text{as } n \to \infty$$

uniformly on $\mathbb{T} \times \mathbb{R}^d$. Therefore, the stochastic integral term in (B.1) converges to the same term with $U^{(n)}$ replaced with the limiting U—i.e., it converges to the

corresponding term in $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$—as $n \to \infty$, for

$$
\mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^t \mathbb{K}_{t-s;x,y}^{\text{BTBM}^d}\left(a(U(s,y)) - a(U^{(n)}(s,y))\right)\mathscr{W}(ds \times dy)\right]^2
$$

$$
\leq C\int_{\mathbb{R}^d}\int_0^t\left[\mathbb{K}_{t-s;x,y}^{\text{BTBM}^d}\right]^2\mathbb{E}\left[U(s,y) - U^{(n)}(s,y)\right]^2 ds dy \longrightarrow 0
$$

as $n \to \infty$. It follows that U satisfies the BTP SIE $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$. Also, the solution is strong since the $U^{(n)}$ are constructed for a given white noise $\mathscr{W}$, and the limit U satisfies (1.3) with respect to that same $\mathscr{W}$. Clearly U is $L^p(\Omega)$ bounded on $\mathbb{T} \times \mathbb{R}^d$, $1 \leq d \leq 3$, for any $p \geq 2$ and for any $T > 0$.

To show uniqueness let $1 \leq d \leq 3$, let $T > 0$ be fixed but arbitrary, and let U_1 and U_2 be two solutions to the BTP SIE (1.3) that are $L^2(\Omega)$-bounded on $\mathbb{T} \times \mathbb{R}^d$. Fix an arbitrary $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Let $D(t, x) = U_2(t, x) - U_1(t, x)$, $L_2(t, x) = \mathbb{E}D^2(t, x)$, and $L_2^*(t) = \sup_{x \in \mathbb{R}^d} L_2(t, x)$ (which is bounded on $\mathbb{T}$ by hypothesis). Then, using (1.3), the Lipschitz condition in (Lip), and taking the supremum over the space variable and using the computation on p. 531 of [1] we have

$$
L_2(t, x) = \int_{\mathbb{R}^d}\int_0^t \mathbb{E}\left[a(U_2(s,y)) - a(U_1(s,y))\right]^2\left[\mathbb{K}_{t-s;x,y}^{\text{BTBM}^d}\right]^2 ds dy
$$

(B.4)
$$
\leq C\int_{\mathbb{R}^d}\int_0^t L_2(s,y)\left[\mathbb{K}_{t-s;x,y}^{\text{BTBM}^d}\right]^2 ds dy
$$

$$
\leq C\int_0^t L_2^*(s)\int_{\mathbb{R}^d}\left[\mathbb{K}_{t-s;x,y}^{\text{BTBM}^d}\right]^2 dy ds \leq C\int_0^t \frac{L_2^*(s)}{(t-s)^{\frac{d}{4}}} ds
$$

Iterating and interchanging the order of integration we get

(B.5)
$$
L_2(t, x) \leq C\left\{\int_0^t L_2^*(r)\left(\int_r^t \frac{ds}{(t-s)^{d/4}(s-r)^{d/4}}\right)dr\right\}
$$

$$
\leq C\left(\int_0^t L_2^*(s)ds\right)
$$

for any $1 \leq d \leq 3$. Hence,

(B.6)
$$
L_2^*(t) \leq C\left(\int_0^t L_2^*(s)ds\right)
$$

for every $t \geq 0$. An easy application of Gronwall's lemma gives that $L_2^* \equiv 0$. So for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $1 \leq d \leq 3$ we have $U_1(t, x) = U_2(t, x)$ with probability one. The indistinguishability of U_1 from U_2 follows from their Hölder regularity, which in turn follows from Remark 2.3. $\qquad\square$

Appendix C. BTP SPDE Kernel Formulation, brief remarks, and a converse to Lemma 1.2

Our main interest in this paper is in the BTP SIE $e_{\text{BTP}}^{\text{SIE}}(a, u_0)$, but—for the reader's convenience—we show here that the spatially-discretized version of the

SPDE (1.5) given by

$$\text{(C.1)} \qquad \begin{cases} \dfrac{d\tilde{U}_n^x(t)}{dt} = \dfrac{\Delta_n u_0(x)}{\sqrt{8\pi t}} + \dfrac{\Delta_n^2 \tilde{U}_n^x(t)}{8} + a(\tilde{U}_n^x(t)) \dfrac{dW_n^x(t)}{\delta_n^{d/2} dt}; \\ \tilde{U}_n^x(0) = u_0(x); \end{cases}$$

is different from the BTRW SIE $e_{\text{BTRW}}^{\text{SIE}}(a, u_0, n)$, and we give its kernel formulation. The derivation, which is similar to the proof of Lemma 1.2, is now sketched. The integral form of (C.1) is of course

$$\text{(C.2)} \qquad \begin{aligned} \tilde{U}_n^x(t) &= u_0(x) + \int_0^t \left[\left(\frac{\Delta_n u_0(x)}{\sqrt{8\pi s}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^x(s) \right) ds + a(\tilde{U}_n^x(s)) \frac{dW_n^x(s)}{\delta_n^{d/2}} \right]; \\ &(t, x) \in \mathbb{T} \times \mathbb{X}_n^d, \ n \in \mathbb{N}, \ \text{a.s. } \mathbb{P}. \end{aligned}$$

We fix an arbitrary $(n, t, x) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{X}_n^d$. Assume that $\tilde{U}_n$ is a semimartingale (in t) satisfying (C.2). Applying Itô's formula and proceeding as in the proof of Lemma 1.2 we get

$$\begin{aligned} \tilde{U}_n^x(t) - \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\text{BTRW}_{\delta_n}^d} u_0(y) &= \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right] \\ &\quad - \int_0^t \sum_{y \in \mathbb{X}_n^d} \left[\frac{\Delta_n \mathbb{K}_{0;x,y}^{\text{BTRW}_{\delta_n}^d}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} \right] \tilde{U}_n^y(s) ds \\ &\quad + \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} \left[\frac{\Delta_n u_0(y)}{\sqrt{8\pi s}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^y(s) \right] ds \\ &= \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} \frac{\Delta_n u_0(y)}{\sqrt{8\pi s}} ds - \int_0^t \frac{\Delta_n \tilde{U}_n^x(s)}{\sqrt{8\pi(t-s)}} ds \\ &\quad + \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right] \end{aligned}$$

I.e, a solution to the SDDE system in equation (C.1) satisfies the Brownian-time random walk (BTRW) kernel (density function) formulation:

$$\text{(C.3)} \qquad \begin{aligned} \tilde{U}_n^x(t) &= \sum_{y \in \mathbb{X}_n^d} \left[\mathbb{K}_{t;x,y}^{\text{BTRW}_{\delta_n}^d} u_0(y) + \Delta_n u_0(y) \int_0^t \frac{\mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d}}{\sqrt{8\pi s}} ds \right] \\ &\quad - \int_0^t \frac{\Delta_n \tilde{U}_n^x(s)}{\sqrt{8\pi(t-s)}} ds + \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BTRW}_{\delta_n}^d} a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}}, \end{aligned}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{X}_n^d$, where the BTRW density is given by (1.7).

Several observations on the special form of (C.3) are in order here. First, unlike second order SPDEs on the lattice (RD, Burgers, etc. . .) and unlike other fourth order SPDEs on the lattice (e.g. see [4, 6] for equations like KS, CH, SH, etc. . .) and unlike the BTRW SIEs in (1.11), the BTRW kernel formulation (C.3) involves terms (the second and third terms on the r.h.s. of (C.3)) that are simply not there in other standard Green function type formulations. This is due to the unique form of the BTP PDE and its discretized version, involving the initial data u_0 in the equation

itself. Second, the third term on the r.h.s. of (C.3) involves the discrete Laplacian of the solution $\Delta_n \tilde{U}_n^x(t)$ with no kernel term, which is fine on the lattice $\mathbb{X}_n^d$, but the continuous-space Laplacian $\Delta U(t, x)$ of the solution to the BTP SPDE $e_{\mathrm{BTP}}^{\mathrm{SPDE}}(a, u_0)$ is not defined in the classical sense. This is why we regard the BTP SPDE as a degenerate version of the BTP SIE and its associated paremetrized BTP SPDE. So, with an eye on the limiting SPDE $e_{\mathrm{BTP}}^{\mathrm{SPDE}}(a, u_0)$, one way to start addressing this difficulty is by reformulating (C.3) into a weaker test function formulation which we now give. Multiplying (C.3) by δ_n and $\xi \in C_c^2(\mathbb{R}^d; \mathbb{R})$, summing over x, and using summation by parts on the third term on the r.h.s. of (C.3), we obtain

$$
\begin{aligned}
(\mathrm{C.4}) \quad \tilde{U}_n^\xi(t) := \sum_{x \in \mathbb{X}_n^d} \tilde{U}_n^x(t)\xi(x)\delta_n^d &= \sum_{x \in \mathbb{X}_n^d} \xi(x) \left\{ \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y) \right\} \delta_n^d \\
&+ \sum_{x \in \mathbb{X}_n^d} \xi(x) \left\{ \sum_{y \in \mathbb{X}_n^d} \Delta_n u_0(y) \int_0^t \frac{\mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d}}{\sqrt{8\pi s}} ds \right\} \delta_n^d \\
&- \sum_{x \in \mathbb{X}_n^d} \left\{ \int_0^t \frac{\tilde{U}_n^x(s)}{\sqrt{8\pi(t-s)}} ds \right\} \Delta_n \xi(x)\delta_n^d \\
&+ \sum_{x \in \mathbb{X}_n^d} \xi(x) \left\{ \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right\} \delta_n^d
\end{aligned}
$$

Here again it is interesting to observe that the form of (C.4) is special, for it has the unusual feature of having a mix of both kernel and test function terms in the same equation, with the third term on the r.h.s. of (C.4) involving only the test function ξ with no kernel terms. It is clear by the derivation of (C.4) above that a solution to $e_{\mathrm{BTRW}}^{\mathrm{SDDE}}(a, u_0, n)$ will satisfy (C.4) for every $\xi \in C_c^2(\mathbb{R}^d; \mathbb{R})$.

   We end with a converse to Lemma 1.2

Lemma C.1. *Assume that for each fixed $(t, x, y) \in \mathbb{R}_+ \times \mathbb{X}_n^{2d}$ the process $\tilde{U}_n$ is a continuous semimartingale in s having the form*

$$
(\mathrm{C.5}) \qquad\qquad \tilde{U}_n^{x,y}(s, t) = u_0(y) + V_n^{x,y}(s, t) + M_n^y(s),
$$

where V_n is the process of bounded variation on compacts (in s) and M_n is the local martingale (in s) in the decomposition of the semimartingale $\tilde{U}_n$. Assume further that $\tilde{U}_n^x(t) := \tilde{U}_n^{x,x}(t, t)$ satisfies (1.11), and that for any fixed pair (t, x), $\mathbb{E} \left| \tilde{U}_n^{x,y}(s, t) \right|^2 \le C$ for all $(s, y) \in [0, t] \times \mathbb{X}_n^d$ for some constant $C > 0$. Then $\tilde{U}_n$ satisfies (1.24).

Remark C.1. Again, the moment boundedness condition above is for convenience and may be relaxed.

Sketch of the proof.   Assume that the (t, x)-parametrized random field $\tilde{U}_n^{x,y}(s, t)$ on $[0, t] \times \mathbb{R}_+ \times \mathbb{X}_n^{2d}$ is a continuous semimartingale in s such that $\tilde{U}_n^x(t) = \tilde{U}_n^{x,x}(t, t)$ satisfies the BTRW SIE (1.11), and assume $\tilde{U}_n^{x,y}(0, t) = u_0(y)$ for all $(t, x, y) \in$

$\mathbb{R}_+ \times \mathbb{X}_n^{2d}$; then as in the proof of Lemma 1.2 we have by Itô's rule and Lemma 1.1

$$\int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} \right] = \tilde{U}_n^x(t) - \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\mathrm{BTRW}_{\delta_n}^d} u_0(y)$$

$$= -\int_0^t \sum_{y \in \mathbb{X}_n^d} \left[\frac{\Delta_n \mathbb{K}_{0;x,y}^{\mathrm{BTRW}_{\delta_n}^d}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \right] \tilde{U}_n^{x,y}(s,t) ds$$

$$+ \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} d\tilde{U}_n^{x,y}(s,t)$$

which is

$$= -\int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[\frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^{x,y}(s,t) \right] ds$$

$$+ \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} dV_n^{x,y}(s,t) + \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} dM_n^y(s)$$

for every $(t,x) \in \mathbb{R}_+ \times \mathbb{X}_n^d$. It is then straightforward to see that

$$\int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}} - dM_n^y(s) \right]$$

(C.6)
$$= -\int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} \left[\frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^{x,y}(s,t) \right] ds$$

$$+ \int_0^t \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t-s;x,y}^{\mathrm{BTRW}_{\delta_n}^d} dV_n^{x,y}(s,t)$$

and that $dM_n^y(s) = a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}}$; i.e.,

$$d\tilde{U}_n^{x,y}(s,t) = \left[\frac{\Delta_n \tilde{U}_n^{x,y}(s,t)\big|_{y=x}}{\sqrt{8\pi(t-s)}} + \frac{1}{8} \Delta_n^2 \tilde{U}_n^{x,y}(s,t) \right] ds + a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}}.$$

We are now done. $\quad\square$

References

[1] Allouba, H. (2006). A Brownian-time excursion into fourth-order PDEs, linearized Kuramoto-Sivashinsky, and BTP-SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$. *Stoch. Dyn.* 6 no. 4, 521–534. MR2285514

[2] Allouba, H. (2003). A linearized Kuramoto-Sivashinsky PDE via an imaginary-Brownian-time-Brownian-angle process. *C. R. Math. Acad. Sci. Paris* 336, no. 4, 309–314. MR1976309

[3] Allouba, H. Kuramoto-Sivashinsky-type SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ via the imaginary-Brownian-time-Brownian-angle representation. In final preparation.

[4] Allouba, H. and Langa, J. A. Nonlinear Kuramoto-Sivashinsky type SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ and their attractors. In preparation.

[5] Allouba, H. and Xiao, Y. BTP SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: modulus of continuity, hitting probabilities, Hausdorff dimensions, and d-dependent variation. In preparation.

[6] Allouba, H. and Duan, J. Swift-Hohenberg SPDEs driven on $\mathbb{R}_+ \times \mathbb{R}^d$ and their attractors. In preparation.

[7] Allouba, H. (2003). SDDEs limits solutions to sublinear reaction-diffusion SPDEs. *Electron. J. Differential Equations.* No. 111, 21 pp. (electronic). MR2011584

[8] Allouba, H. (2002). Brownian-time processes: the PDE connection II and the corresponding Feynman-Kac formula. *Trans. Amer. Math. Soc.* 354, no. 11, 4627–4637 (electronic). MR1926892

[9] Allouba, H. and Zheng, W. (2001). Brown ian-time processes: the PDE connection and the half-derivative generator. *Ann. Probab.* 29, no. 4, 1780–1795. MR1880242

[10] Allouba, H. (2000) Uniqueness in law for the Allen-Cahn SPDE via change of measure. *C. R. Acad. Sci. Paris Sr. I Math.* 330, no. 5, 371–376. MR1751673

[11] Allouba, H. (1998). A non-nonstandard proof of Reimers' existence result for heat SPDEs. *J. Appl. Math. Stochastic Anal.* 11, no. 1, 29–41. MR1613539

[12] Bass, R. (1995). Probabilistic Techniques in Analysis. Springer, Berlin Heidelberg New York. MR1329542

[13] Bass, R (1997). Diffusions and Elliptic Operators. Springer, Berlin Heidelberg New York. MR1483890

[14] Bass, R. and Huili T. The martingale problem for a class of stable-like processes. arXiv:0709.3082v1

[15] Burdzy, K. (1993). Some path properties of iterated Brownian motion. Seminar on Stochastic Processes 1992, Birkhäuser, 67–87. MR1278077

[16] Burdzy, K. (1994). Variation of iterated Brownian motion. Workshop and conference on measure valued processes, stochastic PDEs and interacting particle systems. CRM Proceedings and Lecture Notes 5, 35–53. MR1278281

[17] Dalang, R. (1999). Extending martingale measure stochastic integrals with applications to spatially homogeneous SPDEs. Electron. J. Probab. 4 1–29. MR1684157

[18] Dalang, R. and Sanz-Solé, M. (2005). Regularity of the sample paths of a class of second-order spde's. J. Funct. Anal. 227, no. 2, 304–337. MR2168077

[19] Dalang, R. and Sanz-Solé, M. (2005). Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension 3. 75 pages. arXiv:math/0512540v1

[20] Da Prato, G. and Debussche, A. (1996) Stochastic Cahn-Hilliard equation. Nonlinear Anal. 26, no. 2, 241–263. MR1359472

[21] DeBlassie, R. (2004). Iterated Brownian motion in an open set. *Ann. Appl. Probab.* 14, no. 3, 1529–1558. MR2071433

[22] Doob, J. (1953). Stochastic processes. John Wiley and Sons. MR0058896

[23] Garcia, N. L. and Kurtz, T. G. (2006) Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat. Am. J. Probab. Math. Stat.* 1, 281–303 (electronic). MR2249658

[24] Ikeda, N. and Watanabe, S. (1989). Stochastic differential equations and diffusions, North-Holland Publishing Company. MR1011252

[25] Karatzas, I. and Shreve, S. (1988). Brownian motion and stochastic calculus. Springer-Verlag. MR0917065

[26] Kurtz, T (2007). The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. Electron. J. Probab. 12, 951–965 (electronic). MR2336594

[27] Kurtz, T. (2000). Particle representations for measure-valued population processes with spatially varying birth rates. Stochastic models (Ottawa, ON, 1998), 299–317, CMS Conf. Proc., 26, Amer. Math. Soc., Providence, RI. MR1765017

[28] Le Gall, J. (1994) A path-valued Markov process and its connections with partial differential equations. First European Congress of Mathematics, Vol. II (Paris, 1992). Progr. Math. 120, Birkhäuser, 185–212. MR1341844

[29] Shiga, T. and Shimizu, A. (1980). Infinite dimensional stochastic differential equations and their applications. J. Math. Kyoto Univ. 20-3, 395–416. MR0591802

[30] Sowers, R. (1998). Short-time geometry of random heat kernels. (English summary) Mem. Amer. Math. Soc. 132, no. 629, viii+130 pp. MR1401494

[31] Sowers, R. (1994). Multidimensional reaction-diffusion equations with white noise boundary perturbations. Ann. Probab. 22 no. 4, 2071–2121. MR1331216

[32] Stroock, D. and Varadhan, S. (2006). Multidimensional diffusion processes. Reprint of the 1997 edition. Classics in Mathematics. Springer-Verlag, Berlin, xii+338 pp. ISBN: 978-3-540-28998-2; 3-540-28998-4 MR2190038

[33] Stroock, D. and Zheng, W. (1997). Markov chain approximations to symmetric diffusions. *Ann. Inst. H. Poincar Probab. Statist.* 33, no. 5, 619–649. MR1473568

[34] Temam, R. Infinite-dimensional dynamical systems in mechanics and physics. Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997. xxii+648 pp. ISBN: 0-387-94866-X MR1441312

[35] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. École d'été de Probabilités de Saint-Flour XIV. Lecture Notes in Math. 1180. Springer, New York. MR0876085

[36] Xiao, Y. (1998). Local times and related properties of multidimensional iterated Brownian motion, *J. Theoret. Probab.* 11, no. 2, 383–408. MR1622577

DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT, OHIO 44242, EMAIL: ALLOUBA@MATH.KENT.EDU